TOROIDAL COMPACTIFICATIONS OF INTEGRAL MODELS OF SHIMURA VARIETIES OF HODGE TYPE

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Abstract. We construct smooth projective toroidal compactifications for the integral canonical models of Shimura varieties of Hodge type constructed by Kisin and Vasiu at primes where the level is hyperspecial. This construction is a consequence of the main result of the paper, which shows, without any unramifiedness conditions on the Shimura datum, that the Zariski closure of a Shimura sub-variety of Hodge type in a Chai-Faltings compactification always intersects the boundary in a relative Cartier divisor. This result also provides a new proof of Y. Morita's conjecture on the everywhere good reduction of abelian varieties (over number fields) whose Mumford-Tate group is anisotropic modulo center. We also construct integral models of the minimal (Satake-Baily-Borel) compactification for Shimura varieties of Hodge type.

Introduction

Shimura varieties of Hodge type. This paper is concerned with constructing good compactifications for integral canonical models of Shimura varieties of Hodge type at primes where the level is hyperspecial. Given a Shimura datum (G, X) equipped with an embedding $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), \mathrm{S}^{\pm})$ into a Siegel Shimura datum, and a suitably small compact open $K \subset G(\mathbb{A}_f)$, the Shimura variety $\mathrm{Sh}_K(G, X)$[1] can be viewed as a parameter space for polarized abelian varieties equipped with level structures and additional Hodge tensors.

If we are in the more familiar PEL setting, these additional Hodge tensors can be chosen to consist of endomorphisms and polarizations. One can then define representable PEL type moduli problems over the reflex field $E = E(G, X)$, and even over a suitable localization of its ring of integers, which recover the moduli interpretation for $\mathrm{Sh}_K(G, X)$ over $\mathbb{C}$, and are thus *canonical* models for $\mathrm{Sh}_K(G, X)$ over E or even its ring of integers; cf. [Del71] for the theory over E, and [Kot92] for the integral theory (when the level at p is hyperspecial). The theory of [Del71] applies more generally to show that Shimura varieties of Hodge type admit canonical models over their reflex fields[2], and Milne has used Deligne's results on absolute Hodge cycles to give these canonical models a modular interpretation; cf. [Mil94].

Example. An important class of Shimura data of Hodge type arises from quadratic forms over $\mathbb{Q}$ of signature $(n+, 2-)$. Suppose that we have a vector space U over $\mathbb{Q}$ equipped with such a quadratic form. Then the group $G = \mathrm{GSpin}(U)$ acts naturally on the Clifford algebra C attached to U. We can equip C with an appropriate symplectic form such that we have an embedding $\mathrm{GSpin}(U) \hookrightarrow \mathrm{GSp}(C)$. Moreover, if we take X to be the space of negative definite oriented 2-planes in $U_{\mathbb{R}}$, then (G, X) is a Shimura datum, and we in fact get an embedding $(G, X) \hookrightarrow (\mathrm{GSp}(C), \mathrm{S}^{\pm})$ of Shimura data. This is the **Kuga-Satake construction**; cf. [Del72]. It is important, for example, in the study of the moduli of K3 surfaces (when $n = 19$). Moreover, the Shimura varieties attached to the Spin group Shimura data play a significant role in S. Kudla's program (cf. [Kud04]) for relating intersection numbers on Shimura varieties with Fourier coefficients of Eisenstein series. (G, X) is not of PEL type if $n \geq 5$.

[1] Unless otherwise specified, we will consider the Shimura variety as a scheme over its reflex field.

[2] We now know that every Shimura variety admits such a canonical model; cf. [Mil90].

Compactifications in the case of hyperspecial level. Unfortunately, since Hodge cycles are still transcendentally defined, there is no natural way to use them to obtain a modular interpretation over the ring of integers of E. So, to get a good integral model for $\mathrm{Sh}_K(G,X)$, we have to resort to more *ad hoc* methods. Suppose that the level at p is hyperspecial, and that $v|p$ is a place of E above p. Suppose also that $p>2$. Then, in [Kis10], Kisin constructed[3] the integral canonical model $\mathscr{S}_K(G,X)_{\mathscr{O}_{E,(v)}}$ for $\mathrm{Sh}_K(G,X)$ over the localization of $\mathscr{O}_E$ at v.

Since one of the main interests in having good integral models of Shimura varieties is to facilitate the computation of their zeta functions, and hence their cohomology, we are led to consider the question of their compactification. Over $\mathbb{C}$, Mumford and his collaborators (cf. [AMRT10]) constructed good, toroidal compactifications in the general setting of arithmetic quotients of hermitian symmetric domains. In [Har89] and [Pin90] these compactifications are constructed for Shimura varieties in their natural adelic setting. All these constructions depend on a choice of a certain cone decomposition Σ, called a smooth complete admissible rppcd (cf. (4.2) for the terminology). Given such a choice they produce a smooth compactification $\mathrm{Sh}^{\Sigma}_K(G,X)$ of the Shimura variety $\mathrm{Sh}_K(G,X)$. Our main result is the following theorem; cf. (4.6.13) for a more precise statement.

Theorem 1. *Suppose again that $p>2$. There is a co-final collection of smooth complete admissible rppcds Σ for (G,X,K) such that the integral canonical model $\mathscr{S}_K \coloneqq \mathscr{S}_K(G,X)_{\mathscr{O}_{E,(v)}}$ admits a smooth toroidal compactification $\mathscr{S}^{\Sigma}_K$ that is a proper integral model of $\mathrm{Sh}^{\Sigma}_K(G,X)$. In particular, étale locally around any point, the embedding $\mathscr{S}_K \subset \mathscr{S}^{\Sigma}_K$ is isomorphic to a torus embedding $T \subset \overline{T}$, and the boundary $\mathscr{S}^{\Sigma}_K \backslash \mathscr{S}_K$ is a relative normal crossings divisor over $\mathscr{O}_{E,(v)}$ that admits a stratification parameterized by a conical complex that can be described explicitly in terms of the Shimura datum (G,X,K) and the rppcd Σ.*

The original construction of such integral toroidal compactifications is due to Chai and Faltings ([FC90]) in the case of the Siegel Shimura datum. Their methods were amplified and extended to the case of Shimura varieties of PEL type by K.-W. Lan in [Lan08] (cf. also [Rap78] and [Lar92]). Our method of proof takes as input the existence of the Chai-Faltings compactifications, as well as the compactifications in characteristic 0 mentioned above. As such, it makes essential use of the compatibility between arithmetic and analytic compactifications proven in [Lan10a].

In [Kis10], Kisin also constructs integral canonical models for Shimura varieties of abelian type. It should be possible to extend his method to also construct compactifications for these models; for certain special cases, involving orthogonal Shimura varieties, cf. [MP12b].

The restriction $p>2$ is entirely because of a similar restriction in [Kis10]; cf. (4.6.5) for a discussion of this.

Transversality. A result along the lines of Theorem 1, in the more general setting of Pink's mixed Shimura varieties, has been stated in [Hör10]. This, however, is contingent on a crucial conjecture ([Hör10, 3.3.2]), which we prove along the way. In fact, it (or, rather, a strengthened form of it) is the key to the whole construction. Let us now describe it. We direct the reader to (4.4.7) for a more precise version of what follows.

We now drop the condition that G is unramified at p, and also the condition $p>2$. Suppose that we have an embedding $(G,X) \hookrightarrow (\mathrm{GSp}(V), \mathrm{S}^{\pm}(V))$ into a Siegel Shimura datum. For any compact open $K' \subset \mathrm{GSp}(\mathbb{A}_f)$ with K'_p hyperspecial, the integral model $\mathscr{S}_{K'}(\mathrm{GSp}, \mathrm{S}^{\pm})$ over $\mathbb{Z}_{(p)}$ admits a toroidal compactification $\mathscr{S}^{\Sigma'}_{K'}(\mathrm{GSp}, \mathrm{S}^{\pm})$ constructed by Chai-Faltings [FC90], such that the boundary is a relative Cartier divisor over $\mathbb{Z}_{(p)}$.

[3] We note that a construction due to Vasiu can be found in [Vas99]

Theorem 2. *Suppose K' is such that, with $K = G(\mathbb{A}_f) \cap K'$, the induced map of Shimura varieties*

$$\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})_E$$

is a closed embedding. Then the Zariski closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{S}^{\Sigma'}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})_{\mathscr{O}_{E,(v)}}$ intersects the boundary in a relative Cartier divisor over $\mathscr{O}_{E,(v)}$.

Let us try to explain the main idea behind the proof. Denote the Zariski closure by $\overline{\mathscr{S}}_K$. The result is a local one and can be proven by considering the formal neighborhood of a closed point x_0 in $\overline{\mathscr{S}}_K$. For simplicity, we assume now that x_0 is a closed stratum in $\mathscr{S}^{\Sigma'}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$. In this situation, for any finite extension $L/\mathbb{Q}_p$, and any lift $x \in \mathrm{Sh}_K(G,X)(L)$ of x_0, the fiber A_x of the universal abelian scheme will admit a rigid analytic uniformization $T^{\mathrm{an}}/\iota(Y) \xrightarrow{\simeq} A_x^{\mathrm{an}}$, where $T^{\mathrm{an}} = \underline{\mathrm{Hom}}(X,\mathbb{G}_m^{\mathrm{an}})$ is a split rigid analytic torus with character group X; Y is a free abelian group of 'periods' with $\mathrm{rk}\, Y = \mathrm{rk}\, X$; and $\iota : Y \to T^{\mathrm{an}}$ is a map of analytic groups.

Set $V_x = H^1_{\text{ét}}(A_{x,\overline{L}},\mathbb{Q}_p)$; then the uniformization endows V_x with a three-step weight filtration $W_\bullet V_x$ with $W_0 V_x = W_1 V_x = \mathrm{Hom}(Y,\mathbb{Z}_p)$, and $\mathrm{gr}^W_2 V_x = X \otimes \mathbb{Z}_p$.

Let R be the complete local ring of $\mathscr{S}^{\Sigma'}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$ at x_0; it is a completed torus embedding over W for a split torus $\mathbf{E}$, whose co-character group $\mathbf{B}$ is a lattice in $B(Y \otimes \mathbb{Q})$, the space of symmetric bi-linear pairings on $Y \otimes \mathbb{Q}$. Let $U \subset \mathrm{GSp}(V_x)$ be the unipotent sub-group associated with the weight filtration $W_\bullet V_x$; then $\mathrm{Lie}\, U \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is identified with $B(Y \otimes \mathbb{Q}_p) = \mathbf{B} \otimes \mathbb{Q}_p$.

Choose tensors $\{s_\alpha\} \subset V^{\otimes}$ whose point-wise stabilizer is G. These give rise to Galois-invariant tensors $\{s_{\alpha,\text{ét},x}\} \subset V_x^{\otimes}$. Let $G_x \subset \mathrm{GSp}(V_x)$ be the point-wise stabilizer of the tensors $\{s_{\alpha,\text{ét}}\}$, and let $U_G = U \cap G_x$. Let R_G be the quotient of R corresponding to the completed torus embedding for the sub-torus $\mathbf{E}_G \subset \mathbf{E}$ with co-character group $\mathbf{B}_G = \mathbf{B} \bigcap (\mathrm{Lie}\, U_G \otimes \mathbb{Q}_p)$. The theorem is proven in this case by showing that the geometrically irreducible component of an analytic neighborhood of x in $\overline{\mathscr{S}}_K$ can be identified with an appropriate translate of the analytification of $\mathrm{Spf}\, R_G$.

One non-obvious thing here is the fact that R_G has the right dimension. This is equivalent to showing that $\mathbf{B}_G \otimes \mathbb{Q}$ generates $\mathrm{Lie}\, U_G \otimes \mathbb{Q}_p$ as a $\mathbb{Q}_p$-vector space; but *a priori* it is not even clear that $\mathbf{B}_G$ is non-zero! Our approach to this problem is as follows: For any other lift $x' \in \mathrm{Sh}_K(G,X)(L')$ of x_0 lying in the same irreducible component, the associated semi-stable abelian variety $A_{x'}$ comes equipped with a monodromy pairing $N_{x'} : Y \times X \to \mathbb{Q}$. We show that this pairing gives rise to an element in $\mathbf{B}_G \otimes \mathbb{Q}$. The key here is the existence of a global family of horizontal Hodge tensors over $\mathrm{Sh}_K(G,X)$ arising from $\{s_\alpha\}$, and the compatibility between $N_{x'}$ and the nilpotent operator N on the weakly admissible (φ,N)-module attached to the p-adic Galois representation $H^1(A_{x',\overline{L}'},\mathbb{Q}_p)$. As x' ranges over all lifts of x_0, a simple dimension counting argument shows that the monodromic elements $N_{x'}$ generate all of $\mathrm{Lie}\, U_G \otimes \mathbb{Q}_p$.

Chai-Faltings compactifications at places of bad reduction. The results above have the following application: Consider the moduli stack $\mathbf{M}_{g,p}$ (over $\mathbb{Z}$) of g-dimensional abelian schemes equipped with a polarization of degree p^2. Via Zarhin's trick, it admits a map into the moduli stack $\mathbf{M}_{8g,1}$ of principally polarized $8g$-dimensional abelian schemes. Theorem 2 shows that the closure of the image of $\mathbf{M}_{g,p}$ in any toroidal compactification of $\mathbf{M}_{8g,1}$ intersects the boundary transversally. This allows us to prove:

Theorem 3. *The toroidal compactifications of Chai-Faltings-Lan* [FC90, Lan08] *of $\mathbf{M}_{g,p}$ over $\mathbb{Z}\left[\frac{1}{2p}\right]$ can be extended over $\mathbb{Z}\left[\frac{1}{2}\right]$, and these extensions have the expected properties.*

We refer the reader to (4.5.13) for details and precision. In fact, given [PZ12] and work in progress due to Kisin-Pappas, we expect that the methods of this paper will apply to construct compactifications of most Shimura varieties of Hodge type at places where the level is parahoric.

We note that there is already a compactification over $\mathbb{Z}$ of $\mathbf{M}_{g,p}$ (and in fact of all moduli spaces $\mathbf{M}_{g,d}$ for arbitrary d) available via the work of Alexeev-Nakamura [AN99], Alexeev [Ale02] and Olsson [Ols08]. It is canonical in a very precise sense and has a natural moduli interpretation. However, as observed in the introduction to [Lan08], since it is attached to a specific cone decomposition, it seems ill-suited for the study of Hecke actions and other arithmetic information.

Morita's conjecture. Theorem 2 also has the following pleasant consequence (cf. (4.4.9) in the body of the paper):

Theorem 4. *Suppose that A is an abelian variety defined over a number field F, and suppose that its Mumford-Tate group is anisotropic modulo its center. Then, for every finite place $v|p$ of F, A has potentially good reduction over F_v.*

The hypothesis on the Mumford-Tate group ensures that A does not 'degenerate in characteristic 0'. The theorem says that this is enough to keep it from degenerating in finite characteristic as well. This result gives a different proof of Y. Morita's conjecture (see [Mor75]). Related results can be found in [Pau04],[Vas08] and [Lan10c], with a proof of the full conjecture appearing in [Lee12]. The first two papers, as part of their hypotheses, impose certain local conditions on G. In [Lan10c], Lan also proves the full conjecture as long as A appears in the family of abelian varieties over a compact Shimura variety of PEL type. Finally, Lee proves the full conjecture in [Lee12] using results of [Pau04, Vas08]. Our proof is independent of all these efforts, and applies uniformly without any consideration of special cases.

The minimal compactification. The toroidal compactifications of Mumford, *et. al.* are resolutions of the minimal or Baily-Borel-Satake compactification, which is important from the automorphic perspective, since its L^2 or intersection cohomology is intimately related with the discrete automorphic spectrum of G; cf. [Mor10]. Using by now standard techniques (cf. [FC90, §V.2],[Lan08, §7.2],[Cha90]), we can construct the integral model for the minimal compactification via the Proj construction applied to a certain graded ring of automorphic forms on $\mathscr{S}_K^{\Sigma}$. This gives us the following theorem (cf. 4.8.11):

Theorem 5. *Suppose again that G is unramified at p with $p > 2$ and that K_p is hyperspecial. Then the minimal compactification of $\mathrm{Sh}_K(G,X)$ admits a proper, normal model $\mathscr{S}_K^{\min}$ over $\mathscr{O}_{E,(v)}$ that is stratified by quotients by finite groups of integral canonical models of Shimura varieties of Hodge type. Moreover, the Hecke action of $G(\mathbb{A}_f^p)$ on $\mathscr{S}_K$ extends naturally to an action on $\mathscr{S}_K^{\min}$. Given a complete admissible rppcd Σ as in Theorem 1, there exists a unique map $p_\Sigma : \mathscr{S}_K^{\Sigma} \to \mathscr{S}_K^{\min}$ that extends the identity on $\mathscr{S}_K$ and is compatible with the stratifications on domain and target.*

Tour of contents. We will now briefly describe the contents of the paper.

In §2, we review certain relevant results about log 1-motifs, their Dieudonné theory, and their connection to families of degenerating abelian varieties. Although the results of this paper could have been stated without reference to log 1-motifs for the most part, we believe that they are best phrased in this language. Moreover, log 1-motifs provide a more geometric interpretation of the weakly admissible (φ, N)-module attached to the first p-adic cohomology group of a semi-stable abelian variety. They also allow us to give a new interpretation of the p-adic comparison and Hyodo-Kato isomorphisms for such an abelian variety, which is important when relating the Hyodo-Kato isomorphism to parallel transport between the de Rham cohomologies of the fibers of a semi-stable family of abelian varieties; cf. (3.2.15).

In §3, we review the construction by Chai and Faltings ([FC90]) of formal local models at the boundary of a toroidal compactification of the moduli space of polarized abelian schemes.

Since we deal with level structures and not-necessarily-principal polarizations, we have elected to work in the more precise notation and setting of the work of K.-W. Lan ([Lan08]). We show that the completions of these local models at any point can be interpreted, in the logarithmic setting, as deformation rings for polarized log 1-motifs. We use this interpretation to write down explicit descriptions of these completions in terms of linear algebraic data, much as was done by Faltings in [Fal99] for deformation rings of p-divisible groups. After this, in (3.3), we find the technical heart of the paper. The key result here is (3.3.20), which essentially shows that the completion of (the normalization of) any Shimura variety of Hodge type at a point of the Chai-Faltings compactification must look like a completed torus embedding over a normal base. The proof is essentially the one sketched below Theorem 2, but is a bit more technical, since it needs to take into account the general case where the abelian part of the reduction is non-trivial.

§4 is where we have our payoff, and it forms the bulk of this paper. We review some results about Shimura varieties and Hodge cycles on abelian varieties, as well as the characteristic 0 theory of toroidal compactifications, followed by the Chai-Faltings compactifications of the moduli of polarized abelian varieties. In (4.4), we present a proof of Theorem 2, and deduce from it Morita's conjecture. Theorems 3 and 1 also follow easily. We finish in (4.8) with the construction of the minimal compactification.

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1. Conventions

(1) All rings and monoids will be commutative, unless otherwise noted.

(2) For any prime p, $|\cdot|_p$ will denote the standard p-adic norm with $|p|_p = p^{-1}$.

(3) If L is a discrete valuation field, then $\mathscr{O}_L$ will denote its ring of integers and $\mathfrak{m}_L \subset \mathscr{O}_L$ its maximal ideal.

(4) We will use the geometric notation for change of scalars. If $f : R \to S$ is a map of rings and M is an R-module, then we will denote the induced S-module $M \otimes_{R,f} S$ by f^*M. If the map f is clear from context, then we will also write M_S for the same S-module.

(5) If $\varphi : R \to R$ is an endomorphism of R, then a **φ-module** over R is an R-module M equipped with a map $\varphi^*M \to M$ of R-modules.

(6) Suppose that R is a ring and suppose that $\mathbf{C}$ is an R-linear tensor category that is a faithful tensor sub-category of Mod_R, the category of R-modules. Suppose in addition that $\mathbf{C}$ is closed under taking duals, symmetric and exterior powers in Mod_R. Then, for any object $D \in \mathrm{Obj}(\mathbf{C})$, we will denote by $D^{\otimes}$ the direct sum of the tensor, symmetric and exterior powers of D and its dual.

(7) We will consistently identify the étale topoi of schemes with the same underlying reduced scheme. In particular, if k is a field and B is a local Artin ring with residue field k, then we will, without comment, consider any étale sheaf over $\mathrm{Spec}\, k$ as a sheaf over $\mathrm{Spec}\, B$.

Contents

2. Preliminaries

2.1. **Logarithmic preliminaries.** We assume that the reader is familiar with the basics of log geometry. References include [Kat89] and [Niz08].

2.1.1. We recall that a **log scheme** is a pair (S, M_S) consisting of a scheme S and an étale sheaf of commutative monoids M_S over S equipped with a map of monoids $\alpha : \mathrm{M}_S \to S$ such that $\alpha^{-1}(\mathscr{O}_S^\times) \to \mathscr{O}_S^\times$ is an isomorphism. If $S = \operatorname{Spec} A$ is affine, then, abusing terminology, we will refer to A as a **log ring** or **log algebra**. If P is an adjective applied to commutative monoids, we will say that (S, M_S) is P if, for all geometric points $\overline{s} \to S$, the monoid $\mathrm{M}_{S,\overline{s}} / \mathscr{O}_{S,\overline{s}}^\times$ is P. Here is a list of such adjectives:

- A monoid P is **cancellative** if the map into its group envelope P^{gp} is injective.
- It is **fine** if it is cancellative and finitely generated
- It is **saturated** if it is cancellative, and if, for any $a \in P^{\mathrm{gp}}$, $a^n \in P$, for some $n \in \mathbb{Z}_{>0}$ if and only if $a \in P$.
- It is **fs** if it is fine and saturated.

Definition 2.1.2. A map of monoids $f : P \to Q$ is **continuous** if an element $a \in P$ is invertible if and only if $f(a)$ is invertible in Q.

A map $f : (S, \mathrm{M}_S) \to (T, \mathrm{M}_T)$ of log schemes is **continuous** if, for every geometric point $\overline{s} \to S$, the map $\mathrm{M}_{T,f(\overline{s})} / \mathscr{O}_{T,f(\overline{s})}^\times \to \mathrm{M}_{S,\overline{s}} / \mathscr{O}_{S,\overline{s}}^\times$ is continuous.

If X is an object over (S, M_S) (for a suitable sense of 'over'), we will allow ourselves to slightly abuse terminology and to refer to X as an object over S.

2.1.3. Let (S, M_S) be an fs log scheme. We have the functor $\mathbb{G}_m^{\log}$ on fs log schemes over (S, M_S) given by

$$\mathbb{G}_m^{\log} : (T, \mathrm{M}_T) \to \Gamma(T, \mathrm{M}_T^{\mathrm{gp}}).$$

For the **Kummer log flat topology** on the category of fs log schemes over (S, M_S) (this is a topology refining the fppf topology on S; cf. [Niz08, 2.13]), $\mathbb{G}_m^{\log}$ is a sheaf of abelian groups [Niz08, 2.22]. Let $S_{\mathrm{fl}}^{\log}$ (resp. $S_{\mathrm{fl}}^{\mathrm{cl}}$) be the Kummer log flat (resp. the classical fppf) site over S. We have a natural morphism of sites $\epsilon : S_{\mathrm{fl}}^{\log} \to S_{\mathrm{fl}}^{\mathrm{cl}}$. For any fppf sheaf H over S, we will denote its pull-back $\epsilon^* H$ over the Kummer log flat site also by H.

The étale sheaf of abelian groups $\mathrm{M}_S^{\mathrm{gp}}$ extends to the fppf sheaf $\epsilon_* \mathbb{G}_m^{\log}$; we will denote this extension also by $\mathrm{M}_S^{\mathrm{gp}}$. We have a short exact sequence of fppf sheaves

$$1 \to \mathscr{O}_S^{\times} \to \mathrm{M}_S^{\mathrm{gp}} \to \left(\mathrm{M}_S^{\mathrm{gp}} / \mathscr{O}_S^{\times}\right) \to 1. \tag{2.1.3.1}$$

For any $n \geq 1$, let μ_n be the sheaf over $S_{\mathrm{fl}}^{\mathrm{cl}}$ of n^{th}-roots of unity; then we have the **Kummer short exact sequence** of Kummer log flat sheaves:

$$1 \to \mu_n \to \mathbb{G}_m^{\log} \xrightarrow{\uparrow n} \mathbb{G}_m^{\log} \to 1. \tag{2.1.3.2}$$

Proposition 2.1.4.

(1) *For any sheaf of abelian groups G over $S_{\mathrm{fl}}^{\mathrm{cl}}$, there exists a natural map*

$$\eta_G : \varinjlim_n \underline{\mathrm{Hom}}(\mu_n, G) \otimes \left(\mathrm{M}_S^{\mathrm{gp}} / \mathscr{O}_S^{\times}\right) \to R^1 \epsilon_* G.$$

(2) *Suppose that S is locally Noetherian. If G is representable and is either smooth or finite flat over S, then η_G is an isomorphism.*

(3) *Suppose that $S = \operatorname{Spec} R$, for a Noetherian local ring R. For any $n \in \mathbb{Z}_{>0}$, there is a natural short exact sequence*

$$0 \to H^1\left(S_{\mathrm{fl}}^{\mathrm{cl}}, \mu_n\right) \to H^1\left(S_{\mathrm{fl}}^{\log}, \mu_n\right) \to \left(\mathbb{Z}/n\mathbb{Z}\right) \otimes M \to 0,$$

where $M = H^0\left(S_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{G}_m^{\log}/\mathbb{G}_m\right)$.

Proof. Statements (1) and (2) are from [Niz08, Theorem 3.12] (the results are originally from the unpublished article [Kata]).

For (3), we use the Leray spectral sequence for the functors ϵ and $H^0\left(S_{\mathrm{fl}}^{\mathrm{cl}}, \text{--}\right)$ to get an exact sequence:

$$0 \to H^1\left(S_{\mathrm{fl}}^{\mathrm{cl}}, \mu_n\right) \to H^1\left(S_{\mathrm{fl}}^{\log}, \mu_n\right) \to H^0\left(S_{\mathrm{fl}}^{\mathrm{cl}}, R^1 \epsilon_* \mu_n\right) = \mathrm{End}_{S_{\mathrm{fl}}^{\mathrm{cl}}}\left(\mu_n\right) \otimes M.$$

All it remains to do is to show that the map

$$H^1\left(S_{\mathrm{fl}}^{\log}, \mu_n\right) \to \mathrm{End}_{S_{\mathrm{fl}}^{\mathrm{cl}}}\left(\mu_n\right) \otimes M = \left(\mathbb{Z}/n\mathbb{Z}\right) \otimes M$$

is surjective. Choose some element $\overline{m}$ in the right hand side, an element $m \in M$ lifting $\overline{m}$, and an element $\widetilde{m} \in H^0\left(S_{\mathrm{fl}}^{\log}, \mathbb{G}_m^{\log}\right)$ lifting it (we can always do this: use the exact sequence (2.1.3.1) and the fact that $H^1\left(S_{\mathrm{fl}}^{\mathrm{cl}}, \mathscr{O}_S^{\times}\right) = 0$). The long exact sequence of cohomology associated with the Kummer sequence (2.1.3.2) gives us a boundary map

$$\partial_n : H^0\left(S_{\mathrm{fl}}^{\log}, \mathbb{G}_m^{\log}\right) \to H^1\left(S_{\mathrm{fl}}^{\log}, \mu_n\right).$$

One can check that $\partial_n(\widetilde{m}) \in H^1\left(S_{\mathrm{fl}}^{\log}, \mu_n\right)$ maps onto $\overline{m} \in \left(\mathbb{Z}/n\mathbb{Z}\right) \otimes M$. ∎

2.2. **Log 1-motifs and log F-crystals.** In this sub-section, we will assume that the reader is familiar with the notion of a bi-extension; cf. [Del74, §10.2], [SGA7I, §VII (2.1)] for details. For the theory of 1-motifs, cf. [Del74, §10] and [ABV05]; for that of log 1-motifs, cf. [KKN08b] and [KT03, §4.6].

For any pair (H, G) of sheaves of groups over a scheme S, we will denote by $\mathbf{1}_{H\times G}$ the trivial $\mathbb{G}_m$-bi-extension of $H \times G$; similarly, $\mathbf{1}^{\log}_{H\times G}$ will denote the trivial $\mathbb{G}_m^{\log}$-bi-extension of $H \times G$.

Definition 2.2.1. A **log 1-motif** Q over an fs log scheme (S, M_S) is a tuple $(B, \underline{Y}, \underline{X}, c, c^\vee, \tau)$, where:

- B is an abelian scheme over S, which we will denote Q^{ab}.
- $\underline{Y}$ and $\underline{X}$ are étale sheaves of free abelian groups of finite rank over S, trivialized over a finite étale cover of S. We will denote them as $Q^{\text{ét}}$ and $Q^{\mathrm{mult},C}$, respectively.
- $c : \underline{Y} \to B$ and $c^\vee : \underline{X} \to B^\vee$ are maps of sheaves of groups over S. We will denote them by c_Q and $c_Q^\vee$, respectively.
- $\tau : \mathbf{1}^{\log}_{\underline{Y}\times\underline{X}} \xrightarrow{\simeq} (c \times c^\vee)^*\mathcal{P}_B^{\log}$ is a trivialization of $\mathbb{G}_m^{\log}$-bi-extensions of $\underline{Y} \times \underline{X}$. We will denote it by τ_Q.

Here, $\mathcal{P}_B$ is the Poincaré $\mathbb{G}_m$-bi-extension of $B \times B^\vee$, and $\mathcal{P}_B^{\log}$ is the associated $\mathbb{G}_m^{\log}$-bi-extension.

A map $\varphi : Q_1 \to Q_2$ of log 1-motifs is a tuple $(\varphi^{\mathrm{ab}}, \varphi^{\text{ét}}, \varphi^{\mathrm{mult},C})$, for $? = \mathrm{ab}, \text{ét}$, $\varphi^? : Q_1^? \to Q_2^?$ is a map of sheaves of groups over S and $\varphi^{\mathrm{mult},C} : Q_2^{\mathrm{mult},C} \to Q_1^{\mathrm{mult},C}$. The tuple satisfies: $c_{Q_2}\varphi^{\text{ét}} = \varphi^{\mathrm{ab}}c_{Q_1}$, $c^\vee_{Q_1}\varphi^{\mathrm{mult},C} = \varphi^{\mathrm{ab},\vee}c^\vee_{Q_2}$, and a certain compatibility between τ_{Q_1} and τ_{Q_2}, for which we direct the reader to [Del74, 10.2.12].

The **dual** $Q^\vee$ of a log 1-motif Q is the tuple $(\left(Q^{\mathrm{ab}}\right)^\vee, Q^{\mathrm{mult},C}, Q^{\text{ét}}, c^\vee, c, \tau^\vee)$, where $\tau^\vee$ is the trivialization of the $\mathbb{G}_m^{\log}$-bi-extension $(c_Q^\vee \times c_Q)^*\mathcal{P}^{\log}_{\left(Q^{\mathrm{ab}}\right)^\vee}$ induced from τ via the symmetricity of the Poincaré bi-extension.

A **polarization** of a log 1-motif Q is a map $\lambda : Q \to Q^\vee$ such that $\lambda^{\mathrm{ab}} : Q^{\mathrm{ab}} \to \left(Q^{\mathrm{ab}}\right)^\vee$ is a polarization, and such that $\lambda^{\text{ét}} : Q^{\text{ét}} \to Q^{\mathrm{mult},C}$ is injective.

There is a canonical **weight filtration** $W_\bullet Q$ of a log 1-motif Q with:

$$W_i Q = \begin{cases} 0, & \text{if } i < -2; \\ (0, 0, Q^{\mathrm{mult},C}, 0, 0, 1), & \text{if } i = -2; \\ (Q^{\mathrm{ab}}, 0, Q^{\mathrm{mult},C}, 0, c_Q^\vee, 1) & , \text{ if } i = -1; \\ Q, & \text{if } i = 0. \end{cases}$$

A **1-motif** Q is a log 1-motif $(B, \underline{Y}, \underline{X}, c, c^\vee, \tau)$, where the trivialization τ arises from a trivialization of the $\mathbb{G}_m$-bi-extension $(c \times c^\vee)^*\mathcal{P}_B$.

Remark 2.2.2. If Q is a 1-motif, then we can think of it, as is done in [Del74, §10], as a two-term complex $[Q^{\text{ét}} \to Q^{\mathrm{sab}}]$, where Q^{sab} is the semi-abelian extension of Q^{ab} classified by $c_Q^\vee$. Something similar is true for log 1-motifs as well: we can think of a log 1-motif Q as a two term complex $[Q^{\text{ét}} \to Q^{\mathrm{sab},\log}]$ of Kummer log flat sheaves over (S, M_S), where $Q^{\mathrm{sab},\log}$ is a logarithmic enhancement of Q^{sab}. This is the point of view taken in [KKN08b]; cf. §2.1 of *loc. cit.* In the case where Q^{sab} is a torus with character group $\underline{X}$, we set

$$Q^{\mathrm{sab},\log} = \underline{\mathrm{Hom}}\big(\underline{X}, \mathbb{G}_m^{\log}\big).$$

In the general case, we take $Q^{\mathrm{sab},\log}$ to be the push-out of the diagram

$$\begin{tikzcd} Q^{\mathrm{mult}} \arrow[r] \arrow[d] & Q^{\mathrm{sab}} \\ Q^{\mathrm{mult},\log} & \end{tikzcd}$$

Definition 2.2.3. Given a log 1-motif $Q = (B, \underline{Y}, \underline{X}, c, c^\vee, \tau)$ over (S, M_S), the $\mathbb{G}_m^{\log}/\mathbb{G}_m$-biextension of $\underline{Y} \times \underline{X}$ induced from $(c \times c^\vee)^*\mathcal{P}_B$ is canonically trivialized, and so the trivialization τ determines a pairing $N_\tau : \underline{Y} \times \underline{X} \to \mathbb{G}^{\log}_{m,S}/\mathbb{G}_{m,S}$. For any geometric point $\overline{s} \to S$, the **monodromy of** Q **at** $\overline{s}$, denoted $N_{\tau,\overline{s}}$ is the induced pairing $\underline{Y}_{\overline{s}} \times \underline{X}_{\overline{s}} \to \mathrm{M}^{\mathrm{gp}}_{S,\overline{s}}/\mathscr{O}^\times_{S,\overline{s}}$.

It is easy to see that a log 1-motif is a classical 1-motif precisely when $N_{\tau,\overline{s}}$ is trivial for all geometric points $\overline{s} \to S$.

Definition 2.2.4. Let Q be a log 1-motif over (S, M_S), thought of as a two-term complex $[Q^{\text{ét}} \to Q^{\mathrm{sab},\log}]$ as in (2.2.2). For any prime p, and any $n \in \mathbb{Z}_{>0}$, the p^n**-torsion** $Q[p^n]$ **of** Q is the Kummer log flat sheaf of groups $H^{-1}\big(\mathrm{cone}(Q \xrightarrow{p^n} Q)\big)$.

The **(log)** p**-divisible group** $Q[p^\infty]$ attached to Q is the Kummer log flat sheaf of groups $\bigcup_n Q[p^n]$. When Q is a classical 1-motif, this is a p-divisible group over S in the classical sense.

There is a perfect pairing $Q[p^\infty] \times Q^\vee[p^\infty] \to \mu_{p^\infty}$ identifying $Q^\vee[p^\infty]$ with the Cartier dual of $Q[p^\infty]$ defined precisely as in [Del74, 10.2.5].

Proposition 2.2.5. *There is a contra-variant exact* ***de Rham realization*** *functor*

$$H^1_{\mathrm{dR}} : \big(\text{log 1-motifs over } S\big) \to \big(\text{filtered locally free } \mathscr{O}_S\text{-modules}\big)$$

satisfying the following properties:

1. *For every log* 1*-motif* Q*, the (descending) filtration* $\mathrm{Fil}^\bullet H^1_{\mathrm{dR}}(Q)$ *is concentrated in degree* 1*, with* $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(Q) = \underline{\mathrm{Lie}}(Q^{\mathrm{sab}})^\vee$.
2. *The restriction of* H^1_{dR} *to the category of* 1*-motifs is the (dual of the) classical de Rham realization functor defined in* [Del74, §10].
3. *If* $Q^{\mathrm{ab}} = 0$*, and* $\underline{Y} = Q^{\text{ét}}$*,* $\underline{X} = Q^{\mathrm{mult},C}$*, then*
$$H^1_{\mathrm{dR}}(Q) = \underline{\mathrm{Hom}}\big(\underline{Y}, \mathscr{O}_S\big) \oplus \underline{X} \otimes \mathscr{O}_S; \;\; \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Q) = \underline{X} \otimes \mathscr{O}_S.$$
4. *For every log* 1*-motif* Q*, there is a canonical perfect pairing* $H^1_{\mathrm{dR}}(Q) \times H^1_{\mathrm{dR}}(Q^\vee) \to \mathscr{O}_S$
5. *Suppose that we have a map* $f : (T, \mathrm{M}_T) \to (S, \mathrm{M}_S)$ *of fs log schemes, whose underlying map of schemes is quasi-separated. For every log* 1*-motif* Q *over* S*, there is a natural isomorphism of* $\mathscr{O}_T$*-modules*
$$f^* H^1_{\mathrm{dR}}(Q) \xrightarrow{\simeq} H^1_{\mathrm{dR}}(f^*Q).$$

Proof. In [Del74], the (co-variant) de Rham realization is defined as the Lie algebra of the universal vector extension $E(Q)$ of the two-term complex attached to Q. For a general log 1-motif Q, viewing $Q/W_{-2}Q$ and Q as two-term complexes, we set

$$E(Q) = E(Q/W_{-2}Q) \times_{Q/W_{-2}Q} Q.$$

Then $H^1_{\mathrm{dR}}(Q) = (\underline{\mathrm{Lie}} E(Q))^\vee$, and $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(Q)$ is the image of $\Omega^1_{Q^{\mathrm{sab}}/S}$ in $H^1_{\mathrm{dR}}(Q)$. That this agrees with the definition for 1-motifs follows from [ABV05, §2.4].

The asserted properties are now immediate from their validity in the case of 1-motifs, including (4), for which cf. [Del74, 10.2.7]. ∎

2.2.6. Fix a prime p. We will assume now that all our log schemes (S, M_S) satisfy one of the following conditions:

- $S = \operatorname{Spec} R$, for R a p-adically complete $\mathbb{Z}_p$-algebra; or
- p is nilpotent in S.

In this situation, one can define the **log crystalline site** $\big((S,\mathrm{M}_S)/\mathbb{Z}_p\big)^{\log}_{\mathrm{cris}}$ as in [Kat89, §5] and can therefore speak of **log crystals** (of locally free $\mathscr{O}^{\log}_{S,\mathrm{cris}}$-modules) over (S,M_S). Let φ be the absolute Frobenius on $(S,\mathrm{M}_S)\otimes\mathbb{F}_p$; then we can also speak of **log F-crystals** over (S,M_S): these consist of a pair $(\mathbb{M},\varphi_{\mathbb{M}})$, where $\mathbb{M}$ is a log crystal over (S,M_S) and $\varphi_{\mathbb{M}}:\varphi^*\mathbb{M}\to\mathbb{M}$ is a map of log crystals. Here, for any logarithmic crystal $\mathbb{M}$ over $\mathscr{O}_{S\otimes\mathbb{F}_p/\mathbb{Z}_p}$, its Frobenius pull-back $\varphi^*\mathbb{M}$ is the logarithmic crystal of $\mathscr{O}^{\log}_{S,\mathrm{cris}}$-modules and its evaluation any object $(U,\mathrm{M}_U)\hookrightarrow(T,\mathrm{M}_T)$ that admits a Frobenius lift φ is simply given by $\varphi^*\big(\mathbb{M}\big((T,\mathrm{M}_T)\big)\big)$.

Definition 2.2.7. A **log Dieudonné F-crystal** over S is a tuple $(\mathbb{M},\varphi_{\mathbb{M}},V_{\mathbb{M}},\operatorname{Fil}^1\mathbb{M}(S))$, where $(\mathbb{M},\varphi_{\mathbb{M}})$ is a log F-crystal over S, and:

- $V_{\mathbb{M}}:\mathbb{M}\to\varphi^*\mathbb{M}$ is a map of logarithmic crystals such that $\varphi_M V_{\mathbb{M}} = p$.
- $\operatorname{Fil}^1\mathbb{M}(S)\subset\mathbb{M}(S)$ is a direct summand (called the **Hodge filtration**) such that
$$\varphi^*\big(\operatorname{Fil}^1\mathbb{M}(S_0)\big) = \ker\big(\varphi_{\mathbb{M}(S_0)}:\varphi^*\mathbb{M}(S_0)\to\mathbb{M}(S_0)\big).$$
Here, $\operatorname{Fil}^1\mathbb{M}(S_0) = \operatorname{Fil}^1\mathbb{M}(S)\otimes_{\mathbb{Z}_p}\mathbb{F}_p\subset\mathbb{M}(S_0)$.

We will usually refer to such a tuple simply as the log Dieudonné F-crystal $\mathbb{M}$.

A log Dieudonné F-crystal is a **Dieudonné F-crystal** if the underlying log F-crystal $\mathbb{M}$ arises from a classical F-crystal over S.

We will write $\mathbf{1}$ for the trivial crystal over S; this is naturally a Dieudonné F-crystal when equipped with $\varphi_{\mathbf{1}} = 1$, and $V_{\mathbb{M}} = p$, with $\operatorname{Fil}^1\mathbf{1}(S)$ set to the zero summand of $\mathbf{1}(S) = S$. The **Tate twist** $\mathbf{1}(1)$ is the Dieudonné F-crystal whose underlying crystal is still the trivial crystal, but $\varphi_{\mathbf{1}(1)} = p$, $V_{\mathbf{1}(1)} = 1$, and $\operatorname{Fil}^1\mathbf{1}(1)(S) = \mathbf{1}(1)(S) = S$.

For any log Dieudonné F-crystal $(\mathbb{M},\varphi_{\mathbb{M}},V_{\mathbb{M}},\operatorname{Fil}^1\mathbb{M}(S))$ over S, its **Cartier dual** is the log Dieudonné F-crystal determined by the tuple
$$(\mathbb{M}^\vee,\varphi_{\mathbb{M}^\vee},V_{\mathbb{M}^\vee},\operatorname{Fil}^1\mathbb{M}^\vee(S)) = (\mathbb{M}^\vee,V_{\mathbb{M}}^\vee,\varphi_{\mathbb{M}}^\vee,\big(\mathbb{M}(S)/\operatorname{Fil}^1\mathbb{M}(S)\big)^\vee).$$
There is a natural perfect pairing $\mathbb{M}\times\mathbb{M}^\vee\to\mathbf{1}(1)$ of log Dieudonné F-crystals.

Proposition 2.2.8. *There is an exact contra-variant functor*
$$\mathbb{D}:\big(\text{log 1-motifs over } S\big)\to\big(\text{log Dieudonné } F\text{-crystals over } S\big)$$
with the following properties:

1. *For every log 1-motif Q, there is a canonical identification of filtered $\mathscr{O}_S$-modules*
$$\big(\mathbb{D}(Q)(S),\operatorname{Fil}^1\mathbb{D}(Q)(S)\big) = \big(H^1_{\mathrm{dR}}(Q),\operatorname{Fil}^1 H^1_{\mathrm{dR}}(Q)\big).$$
2. *If Q is a classical 1-motif, then we have a canonical identification of F-crystals $\mathbb{D}(Q) = \mathbb{D}\big(Q[p^\infty]\big)$, where the right hand side is the classical Dieudonné crystal associated with the p-divisible group $Q[p^\infty]$* [Mes72].
3. *For any log 1-motif Q over S, there is a natural perfect pairing*
$$\mathbb{D}(Q)\times\mathbb{D}(Q^\vee)\to\mathbf{1}(1)$$
identifying $\mathbb{D}(Q^\vee)$ with $\mathbb{D}(Q)^\vee$.
4. *If we have a map $f:(T,\mathrm{M}_T)\to(S,\mathrm{M}_S)$ of fs log schemes, such that the underlying map of schemes is quasi-separated; then, for any log 1-motif Q over T, there is a natural isomorphism*
$$\mathbb{D}(f^*Q)\xrightarrow{\simeq} f^*\mathbb{D}(Q).$$

Proof. A proof of the proposition can be obtained from [MS11, §1.3], which in turn is based on [KT03, §4.7]. The construction proceeds by first considering the case where $Q^{\mathrm{ab}} = 0$, and then uses the fact that every log 1-motif can be built up via push-out and extensions from 1-motifs and log 1-motifs with trivial abelian part. ☐

Remark 2.2.9. By the exactness of $\mathbb{D}$, we observe that $\mathbb{D}(Q)$ inherits a three-step weight filtration from the weight filtration on Q:

$$0 = W_{-1}\mathbb{D}(Q) \subset W_0\mathbb{D}(Q) = \mathbb{D}(Q^{\text{ét}}) \subset W_1\mathbb{D}(Q) = \mathbb{D}\big(\mathrm{gr}^W_{-1} Q\big) \subset W_2\mathbb{D}(Q) = \mathbb{D}(Q).$$

2.3. **Degenerating abelian varieties.**

Definition 2.3.1. A **polarized log 1-motif** over (S, M_S) is a tuple (Q, λ), where Q is a log 1-motif over (S, M_S) and λ is a polarization of Q.

We can also think of a polarized log 1-motif as an 8-tuple $(B, \underline{Y}, \underline{X}, c, c^\vee, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau)$, where τ is such that $(1 \times \lambda^{\text{ét}})^*\tau$ is a *symmetric* trivialization of the symmetric $\mathbb{G}_m^{\log}$-bi-extension $(c \times c^\vee\lambda^{\text{ét}})^*\mathcal{P}_B$ of $\underline{Y} \times \underline{Y}$. This gives us a tuple very much like the ones appearing in the category $\mathrm{DD}_{\mathrm{pol}}$ considered in [FC90, §III.2].

Definition 2.3.2. Let (Q, λ) be a polarized log 1-motif corresponding to a tuple $(B, \underline{Y}, \underline{X}, c, c^\vee, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau)$. We will say that (Q, λ) is **positively polarized** if, for every geometric point $\bar{s} \to S$, the pairing

$$Y \times Y \xrightarrow{1\times\lambda^{\text{ét}}} Y \times X \xrightarrow{N_{\tau,\bar{s}}} \mathrm{M}^{\mathrm{gp}}_{S,\bar{s}} / \mathscr{O}^\times_{S,\bar{s}}$$

is positive definite. Here, we say that an element in $\mathrm{M}^{\mathrm{gp}}_{S,\bar{s}} / \mathscr{O}^\times_{S,\bar{s}}$ is positive if it lies in $\big(\mathrm{M}_{S,\bar{s}} / \mathscr{O}^\times_{S,\bar{s}}\big)\backslash\{1\}$.

A log 1-motif Q is **positively polarizable** if there exists a polarization λ such that (Q, λ) is positively polarized.

2.3.3. Suppose now that $S = \operatorname{Spec} R$, where R is a complete local normal Noetherian ring, and suppose that the log structure on S is defined by a divisor $D \subset S$. Let $U \subset S$ be the complement of D. Let $\mathbf{DEG}_{\mathrm{pol}}(S)$ be the category of positively polarized log 1-motifs (Q, λ) over S. Let $\mathbf{DD}_{\mathrm{pol}}(S)$ be the category of pairs (A, λ), where A is a semi-abelian scheme whose restriction $A|_U$ to U is an abelian scheme over U, and λ is a polarization of $A|_U$.

Theorem 2.3.4. *With the hypotheses as above, there is a functorial (in S) exact equivalence of categories*

$$\mathrm{M}_{\mathrm{pol},S} : \mathbf{DD}_{\mathrm{pol}}(S) \xrightarrow{\simeq} \mathbf{DEG}_{\mathrm{pol}}(S).$$

Moreover, suppose we have $(Q, \lambda) \in \mathbf{DD}_{\mathrm{pol}}(S)$ with $(A, \lambda) = \mathrm{M}_{\mathrm{pol},S}\big((Q, \lambda)\big)$ in $\mathbf{DEG}_{\mathrm{pol}}(S)$. Then:

1. *For every prime ℓ, there is a canonical identification of ℓ-divisible groups $\big(A|_U\big)[\ell^\infty] = Q[\ell^\infty]|_U$, compatible with the Weil pairings induced by the polarizations.*
2. *There is a canonical identification $H^1_{\mathrm{dR}}\big(A|_U\big) = H^1_{\mathrm{dR}}(Q)|_U$ of filtered $\mathscr{O}_U$-modules, compatible with the pairings induced by the polarizations.*

Proof. The construction/proof can be found in [FC90, Ch. III]; cf. also [Lan08, 4.4.16]. Note that when $R = \mathscr{O}_K$ is the ring of integers of a p-adic field, and $D = \operatorname{Spec} k$ is its special point, then the result is due to Raynaud [Ray71]. (1) follows from [Lan08, 4.5.3.10].

Let us show (2). First, suppose that the divisor $D \subset S$ does not contain the reduced subscheme underlying the special fiber $S \otimes \mathbb{F}_p$. If (A, λ) is a polarized abelian scheme over U corresponding to the polarized log 1-motif (Q, λ') over S, then both $H^1_{\mathrm{dR}}(A)$ and $H^1_{\mathrm{dR}}(Q)|_U$ can be naturally identified with the de Rham cohomology of the universal vector extension p-divisible group $A[p^\infty] = (Q|_U)[p^\infty]$ over U.

In general, after a finite étale base change if necessary, we can assume that (Q, λ') corresponds to a tuple $(B, \lambda^{\mathrm{ab}}, Y, X, \lambda^{\text{ét}}, c, c^\vee, \tau)$, where Y and X are constant. Using [Lan08, 6.4.1.1(6)],

one can now show that there exists a complete local ring R' at the boundary of a smooth toroidal compactification of an appropriate moduli space of polarized abelian varieties, equipped with a 'universal' tuple $(B', \lambda^{\mathrm{ab},'}, Y, X, \lambda^{\text{ét}}, c', c^{\vee,'}, \tau')$, and a map $R' \to R$ that gives rise to (Q, λ') over S and the corresponding polarized abelian scheme (A, λ) over U. Since R' is log formally smooth over $\mathbb{Z}_p$, the boundary divisor in $\operatorname{Spec} R'$ does not contain the special fiber of $\operatorname{Spec} R'$, and we conclude now from the previous paragraph. $\square$

2.4. p-adic Hodge theory.

2.4.1. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vectors with coefficients in k equipped with its Frobenius lift φ_W, and let $K_0 = W\left[p^{-1}\right] \subset K$ be the maximal absolutely unramified sub-field. Fix an algebraic closure $\overline{K}$ for K, and let $\widehat{\overline{K}}$ be its completion. Let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K.

Fix a uniformizer $\pi \in K$; let $E(u) \in W[u]$ be the associated monic Eisenstein polynomial, so that we can view $\mathscr{O}_K$ as the quotient $W[u]/(E(u))$. Let S be the p-adic completion of the divided power envelope of the surjection $W[u] \to \mathscr{O}_K$ carrying u to π, and set $\operatorname{Fil}^1 S = \ker(S \to \mathscr{O}_K)$: by construction $\operatorname{Fil}^1 S$ is equipped with divided powers compatible with those on pS. Concretely, we have (cf. [Bre00, 2.1.1]):

$$S = \Big\{ \sum_i a_i \frac{u^i}{q(i)!} \in K_0[|u|] : a_i \in W, \lim_{i \to \infty} a_i = 0 \Big\}.$$

For every $i \in \mathbb{Z}_{\geq 0}$, let $\operatorname{Fil}^i S = (\operatorname{Fil}^1 S)^{[i]}$ be the i^{th} divided power of $\operatorname{Fil}^1 S$.

S is equipped with the log structure induced by the divisor $u = 0$, and the natural log structure on $\mathscr{O}_K$ is induced from this one via the surjection $S \twoheadrightarrow \mathscr{O}_K$.

2.4.2. Consider the category of pairs (A, α) consisting of a p-adically complete W-algebra A, and a surjection $\alpha : A \twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}}$ whose kernel is equipped with divided powers compatible with the canonical divided power structure on pR. By [Fon94a, 2.2.1], this category has an initial object $(A_{\mathrm{cris}}, \theta)$. Put differently, for every $n \in \mathbb{Z}_{>0}$, the PD thickening

$$\operatorname{Spf} \theta_n : \operatorname{Spf} \mathscr{O}_{\overline{K}}/p \hookrightarrow \operatorname{Spec} A_{\mathrm{cris}}/p^n$$

is the final object in the crystalline site $\big((\mathscr{O}_{\overline{K}}/p)/(W/p^n)\big)_{\mathrm{cris}}$.

Concretely, A_{cris} is constructed as follows. Let

$$R = \varprojlim_{x \mapsto x^p} \mathscr{O}_{\overline{K}}/p$$

be the perfection of $\mathscr{O}_{\overline{K}}/p$: it consists of coherent sequences of p-power roots of elements of $\mathscr{O}_{\overline{K}}/p$, and the elements of its underlying multiplicative monoid can also be interpreted as coherent sequences of p-power roots of elements of $\mathscr{O}_{\widehat{\overline{K}}}$. Let $W(R)$ be the ring of Witt vectors with coefficients in R; there is a natural surjection $\theta : W(R) \twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}}$ and A_{cris} is the p-adic completion of the divided power envelope of θ. Note that $\mathscr{G}_K$ naturally acts on everything in sight.

By construction, we have

$$A_{\mathrm{cris}} = \varprojlim_n H^0\Big(\big((\mathscr{O}_{\overline{K}/p})/(W/p^n)\big)_{\mathrm{cris}}, \mathscr{O}_{\mathrm{cris}} \Big),$$

where $\mathscr{O}_{\mathrm{cris}}$ is the crystalline structure sheaf. In particular, A_{cris} is canonically endowed with a Frobenius lift φ that commutes with the $\mathscr{G}_K$-action.

Consider the Teichmüller lift $[\cdot] : R \setminus \{0\} \to A_{\mathrm{cris}} \setminus \{0\}$: it is a map of multiplicative monoids. We also have a natural injection $\mathbb{Z}_p(1) \hookrightarrow R^\times$, where we view an element $\epsilon \in \mathbb{Z}_p(1)$ as a coherent sequence $\underline{\epsilon} \in R^\times$ of p-power roots of unity. Composing this with the

Teichmüller lift gives us an embedding $\mathbb{Z}_p(1) \hookrightarrow A_{\mathrm{cris}}^{\times}$. For any element $[\underline{\epsilon}]$ in its image, the series $\sum_n (-1)^{n-1}(n-1)!([\underline{\epsilon}]-1)^{[n]}$ converges to an element $t(\epsilon) \in A_{\mathrm{cris}}$, and the assignment $\epsilon \mapsto t(\epsilon)$ gives a Galois equivariant embedding $t : \mathbb{Z}_p(1) \hookrightarrow A_{\mathrm{cris}}$. .

2.4.3. Consider the category of pairs (B, β), where B is a p-adically complete log S-algebra and $\beta : B \twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}}$ is a formal log PD thickening. By [Lod07, Theorem 1.1], this category has an initial object $(B^+_{\log}, \beta_{\log})$. Here, $\mathscr{O}_{\widehat{\overline{K}}}$ is viewed as the direct limit of the fs log S-algebras $\mathscr{O}_L$, where L ranges over all finite extensions of K, and so inherits the direct limit log structure; this log structure is not fs, but it is saturated.

This object can also be described explicitly. Let A_{st} be the p-adic completion of the divided power algebra $A_{\mathrm{cris}}\langle X\rangle$ in one variable over A_{cris}. For any choice $\underline{\pi} \in R$ of a coherent sequence of p-power roots of π, we can view A_{st} as an S-algebra via the map $u \mapsto [\underline{\pi}](X+1)^{-1}$. We can also endow it with the log structure associated with the pre-log structure

$$\begin{aligned}(R \setminus \{0\}) \oplus \mathbb{N} &\to A_{\mathrm{st}} \\ (r, i) &\mapsto [r]([\underline{\pi}](X+1)^{-1})^i.\end{aligned}$$

With this log structure, A_{st} can in fact be viewed as a log S-algebra. We now have a surjection

$$\begin{aligned}\theta_{\log} : A_{\mathrm{st}} &\twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}} \\ X &\mapsto 0\end{aligned}$$

of log S-algebras extending $\theta : A_{\mathrm{cris}} \to \mathscr{O}_{\widehat{\overline{K}}}$, and $\ker(\theta_{\log})$ has a natural divided power structure that extends that on $\ker(\theta)A_{\mathrm{st}}$.

Proposition 2.4.4. *There is a canonical isomorphism*

$$(A_{\mathrm{st}}, \theta_{\log}) \xrightarrow{\simeq} (B^+_{\log}, \beta_{\log})$$

of formal log PD thickenings of $\mathscr{O}_{\widehat{\overline{K}}}$ over S.

Proof. This is due to Kato; a proof can be found in [Lod07, Proposition 1.3]; cf. also [Bre97, §2]. □

By construction and (2.4.4) above, we have:

$$A_{\mathrm{st}} = \varprojlim_n H^0\Big(\big((\mathscr{O}_{\overline{K}/p})/(S/p^nS)\big)_{\log\mathrm{cris}}, \mathscr{O}_{\log\mathrm{cris}}\Big),$$

where $\big((\mathscr{O}_{\overline{K}/p})/(S/p^nS)\big)_{\log\mathrm{cris}}$ is the log crystalline site for $\mathscr{O}_{\overline{K}}/p$ over S/p^nS, and $\mathscr{O}_{\log\mathrm{cris}}$ is its structure sheaf. In particular, A_{st} is endowed with a natural $\mathscr{G}_K$-action and a commuting Frobenius lift φ that is compatible with the corresponding structures on A_{cris}. It is also endowed with a logarithmic connection $A_{\mathrm{st}} \to A_{\mathrm{st}} \otimes_{W[u]} W[u]\,\mathrm{dlog}(u)$, for which φ is parallel. Equivalently, it has an S-derivation $\mathcal{N} : A_{\mathrm{st}} \to A_{\mathrm{st}}$ lying over the derivation $u\frac{d}{du}$ of S, and satisfying $\mathcal{N}\varphi = p\varphi\mathcal{N}$. For a concrete description of these structures, cf. [Bre99, 2.2.2]; or [Bre97, §2].

2.4.5. Suppose that we have a log 1-motif Q over $\mathscr{O}_K$. Associated with this is the Tate module $T_p(Q) = \lim_n Q[p^n](\overline{K})$: this is a continuous $\mathscr{G}_K$-representation over $\mathbb{Z}_p$. Set

$$\mathcal{M}(Q) = \varprojlim_n \mathbb{D}(Q)\big(\mathrm{Spec}(\mathscr{O}_K/p\mathscr{O}_K) \hookrightarrow \mathrm{Spec}\, S/p^nS\big).$$

$\mathcal{M}(Q)$ is equipped with the structure of a φ-module over S

$$\varphi_{\mathcal{M}(Q)} : \varphi^*\mathcal{M}(Q) \to \mathcal{M}(Q),$$

and a topologically quasi-nilpotent integrable logarithmic connection ∇. This gives rise to the derivation $\mathcal{N} = \nabla(-u\frac{d}{du})$ on $\mathcal{M}(Q)$. Moreover, there is a canonical identification

$$\mathcal{M}(Q) \otimes_S \mathscr{O}_K = H^1_{\mathrm{dR}}(Q),$$

This gives us the S-sub-module $\mathrm{Fil}^1 \mathcal{M}(Q) \subset \mathcal{M}(Q)$, defined to be the pre-image of the Hodge filtration in $H^1_{\mathrm{dR}}(Q)$: it satisfies $\mathrm{Fil}^1 S \cdot \mathcal{M}(Q) \subset \mathrm{Fil}^1 \mathcal{M}(Q)$, so that $\mathcal{M}(Q)$ has the structure of a filtered module over S. Equip A_{st} with the divided power filtration.

Proposition 2.4.6. *There exists a natural $\mathscr{G}_K$-equivariant map*

$$j_Q : T_p(Q) \to \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^1,\mathcal{N}}\bigl(\mathcal{M}(Q), A_{\mathrm{st}}\bigr)$$

This is an injection with finite cokernel in general and an isomorphism when $p > 2$.

Proof. We can interpret $T_p(Q)$ as the group

$$\mathrm{Hom}\bigl((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathscr{O}_{\widehat{\overline{K}}}}, Q[p^\infty]_{\mathscr{O}_{\widehat{\overline{K}}}}\bigr).$$

Moreover, we have a natural map

$$T_p(Q) \to \mathrm{Hom}_{\varphi,V,\mathrm{Fil}^1}\left(\mathbb{D}\bigl(Q_{\mathscr{O}_{\widehat{\overline{K}}}}\bigr), \mathbb{D}\bigl((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathscr{O}_{\widehat{\overline{K}}}}\bigr)\right).$$

Evaluating the crystals on the thickening $A_{\mathrm{st}} \twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}}$ gives us a natural map

$$j_Q : T_p(Q) \to \mathrm{Hom}_{A_{\mathrm{st}},\varphi,\mathrm{Fil}^1,\mathcal{N}}\left(\mathbb{D}(Q_{\mathscr{O}_{\widehat{\overline{K}}}})(A_{\mathrm{st}}), A_{\mathrm{st}}\right) = \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^1,\mathcal{N}}\bigl(\mathcal{M}(Q), A_{\mathrm{st}}\bigr).$$

If Q is a classical 1-motif, it follows from [Fal99, Theorem 7] and its argument that j_Q is an injection with finite cokernel and that it is an isomorphism when $p > 2$.

In general, the naturality of j_Q implies that it respects the weight filtration $W_\bullet Q$. Since the associated graded for the weight filtration is a classical 1-motif, we see that $\mathrm{gr}^W_\bullet j_Q = j_{\mathrm{gr}^W_\bullet Q}$, and therefore j_Q itself must be an isomorphism. □

2.4.7. We can now define some Fontaine period rings.

- $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[p^{-1}]$: as observed before, it is endowed with a Frobenius lift φ and a $\mathscr{G}_K$-action.
- B^+_{dR} is the $(\ker\theta)$-adic completion of B^+_{cris}: it is a complete DVR with residue field $\widehat{\overline{K}}$; $B_{\mathrm{dR}} = \mathrm{Fr}(B^+_{\mathrm{dR}})$ is its fraction field. B_{dR} is filtered by the powers $\mathrm{Fil}^i B_{\mathrm{dR}}$ of the maximal ideal $\mathrm{Fil}^1 B^+_{\mathrm{dR}}$ of B^+_{dR}. $B^+_{\mathrm{cris}} \otimes_{K_0} K$ embeds naturally in B^+_{dR} and inherits its filtration.
- B^+_{st} is the image of $A_{\mathrm{st}}[p^{-1}]$ in B^+_{dR} under the map of A_{cris}-algebras $X \mapsto \bigl(\frac{\pi}{[\pi]} - 1\bigr)$. It is endowed with a Frobenius lift φ, a $\mathscr{G}_K$-action and a B^+_{cris}-derivation N satisfying $N\varphi = p\varphi N$. Again, $B^+_{\mathrm{st}} \otimes_{K_0} K$ embeds in B^+_{dR} and also inherits a filtration from it. Note that one can define B^+_{st} canonically as a B^+_{cris}-algebra, without making the choice of uniformizer π. The $\mathscr{G}_K$-equivariant embedding $B^+_{\mathrm{st}} \otimes_{K_0} K \hookrightarrow B^+_{\mathrm{dR}}$ of B^+_{cris}-algebras, however, depends on such a choice; cf. [Fon94a] for more details.

2.4.8. Let Q be a log 1-motif over $\mathscr{O}_K$; let Q_0 be the log 1-motif over $k_{\mathbb{N}}$ obtained by reducing Q along the surjection $\mathscr{O}_K \twoheadrightarrow k_{\mathbb{N}}$. Set

$$H^1_{\mathrm{\acute{e}t}}\bigl(Q_{\overline{K}}, \mathbb{Z}_p\bigr) := T_p(Q)^\vee.$$

There is a canonical surjection $W_{\mathbb{N}} \to k_{\mathbb{N}}$ with underlying map of rings $W \to k$ that is obtained as follows: The Teichmüller lift $[\cdot] : k \to W$ is a map of monoids and so we get a canonical map $\mathrm{M}_{k_{\mathbb{N}}} \to k \xrightarrow{[\cdot]} W$. This is the pre-log structure to which the log structure

$\mathrm{M}_{W_{\mathbb{N}}} \to W$ is attached. So we obtain a canonical embedding $\mathrm{M}_{k_{\mathbb{N}}} \hookrightarrow \mathrm{M}_{W_{\mathbb{N}}}$ with quotient $1+pW$. Now the natural inclusion $1+pW \hookrightarrow \mathrm{M}_{W_{\mathbb{N}}}$ gives us a splitting

$$\mathrm{M}_{W_{\mathbb{N}}} = (1+pW) \oplus \mathrm{M}_{k_{\mathbb{N}}}.$$

This gives us the surjection $\mathrm{M}_{W_{\mathbb{N}}} \to \mathrm{M}_{k_{\mathbb{N}}}$ underlying $W_{\mathbb{N}} \to k_{\mathbb{N}}$. Moreover, it also allows us to define a Frobenius lift on $W_{\mathbb{N}}$ that acts as the p-power map on $\mathrm{M}_{k_{\mathbb{N}}}$ and restricts to the canonical Frobenius lift on $1+pW$. Set

$$M_0(Q) \coloneqq \mathbb{D}(Q_0)(W_{\mathbb{N}}) = \varprojlim_n \mathbb{D}(Q_0)\big(\operatorname{Spec} k_{\mathbb{N}} \hookrightarrow \operatorname{Spec}(W/p^nW)_{\mathbb{N}}\big).$$

Note that there is a natural surjection $S \to W_{\mathbb{N}}$ that provides us with an identification of φ-modules

$$\mathcal{M}(Q) \otimes_S W = M_0(Q).$$

Let $N_0 : M_0(Q) \to M_0(Q)$ be the reduction of $\mathcal{N}$.

Lemma 2.4.9. *There is a unique isomorphism*

$$\xi : (M_0(Q) \otimes_W S)\big[p^{-1}\big] \xrightarrow{\simeq} \mathcal{M}(Q)\big[p^{-1}\big] \tag{2.4.9.1}$$

preserving φ and $\mathcal{N}$. Here, we equip $M_0(Q) \otimes_W S$ with the diagonal φ-module structure and the map $N_0 \otimes 1 + 1 \otimes N$.

Proof. See [Bre97, 6.2.1.1] for the construction of ξ; it is a divided power avatar of Dwork's trick. □

Theorem 2.4.10. *Fix a uniformizer $\pi \in \mathscr{O}_K$ as above. Then there exist natural isomorphisms*

$$\beta_{Q,\text{H-K},\pi} : \mathbb{D}(Q_0)(W_{\mathbb{N}}) \otimes_W K \xrightarrow{\simeq} H^1_{\mathrm{dR}}(Q) \otimes_{\mathscr{O}_K} K. \qquad \text{(Hyodo-Kato)} \tag{2.4.10.1}$$

$$\beta_{Q,\mathrm{st},\pi} : H^1_{\text{ét}}\big(Q_{\overline{K}}, \mathbb{Z}_p\big) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{st}} \xrightarrow{\simeq} \mathbb{D}(Q_0)(W_{\mathbb{N}}) \otimes_W B^+_{\mathrm{st}}. \qquad (p\text{-adic comparison}) \tag{2.4.10.2}$$

They satisfy the following properties:

1. *Equip $\mathbb{D}(Q_0)(W_{\mathbb{N}}) \otimes_W K$ with the filtration induced from the Hodge filtration on $H^1_{\mathrm{dR}}(Q)_K$ via the isomorphism $\beta_{Q,\text{H-K},\pi}$. Then $\beta_{Q,\mathrm{st},\pi}$ is $(\varphi, N, \mathscr{G}_K)$-equivariant, and also respects filtrations once we tensor with K.*
2. *If Q is a classical 1-motif, both $\beta_{Q,\text{H-K},\pi}$ and $\beta_{Q,\mathrm{st},\pi}$ are independent of π and agree with the classical comparison isomorphisms obtained from those for the p-divisible group $Q[p^\infty]$ over $\mathscr{O}_K$ (cf.* [Fal99, §6]*).*
3. *Both $\beta_{Q,\text{H-K},\pi}$ and $\beta_{Q,\mathrm{st},\pi}$ are compatible with polarization pairings.*

Proof. We will give a construction for the isomorphisms $\beta_{Q,\text{H-K},\pi}$ and $\beta_{Q,\mathrm{st},\pi}$. The enumerated properties will be clear from the construction, which is a direct extension of [Fal99, Theorem 7].

We obtain (2.4.10.1) simply by reducing (2.4.9.1) from S to $\mathscr{O}_K$. We also have:

$$\begin{aligned} T_p(Q)\left[\frac{1}{p}\right] &\xrightarrow[(2.4.6)]{\simeq} \operatorname{Hom}_{S,\varphi,\mathrm{Fil}^1,\mathcal{N}}\big(\mathcal{M}(Q), A_{\mathrm{st}}\big)\left[\frac{1}{p}\right] \\ &\xrightarrow[(2.4.9.1)]{\simeq} \operatorname{Hom}_{S,\varphi,\mathrm{Fil}^1,\mathcal{N}}\big(\mathbb{D}(Q_0)(W_{\mathbb{N}}) \otimes_W S, B^+_{\mathrm{st}}\big) \\ &= \operatorname{Hom}_{\varphi,\mathrm{Fil}^1,N}(\mathbb{D}(Q_0)(W_{\mathbb{N}}), B^+_{\mathrm{st}}). \end{aligned}$$

Dualizing and tensoring up to B^+_{st} now gives us (2.4.10.2). □

Remark 2.4.11. The construction of the isomorphism $\beta_{Q,\text{H-K},\pi}$ above is along the lines of Deligne's original construction for abelian schemes over $\mathscr{O}_K$; cf. [BO83, 2.9].

We have the following corollary, which we will use without comment in (3.3):

Corollary 2.4.12. *$H^1_{\text{ét}}(Q_{\overline{K}}, \mathbb{Z}_p)$ is a semi-stable Galois representation with weights in $\{0,1\}$. Let*

$$D_{\text{st}}(Q) = \operatorname{Hom}_{\mathscr{G}_K}\big(\mathbb{Q}_p, B_{\text{st}} \otimes_{\mathbb{Z}_p} H^1_{\text{ét}}(Q_{\overline{K}}, \mathbb{Z}_p)\big)$$

be the attached weakly admissible filtered (φ, N)-module. Then there exists a natural isomorphism of fitlered (φ, N)-modules

$$D_{\text{st}}(Q) \xrightarrow{\simeq} \mathbb{D}(Q_0)(W_{\mathbb{N}})[p^{-1}].$$

Moreover, the map $N : D_{\text{st}}(Q) \to D_{\text{st}}(Q)$ factors as:

(2.4.12.1) $D_{\text{st}}(Q) \twoheadrightarrow D_{\text{st}}(Q^{\text{mult}}) = Q^{\text{mult},C} \otimes_{\mathbb{Z}} K_0 \xrightarrow{N_Q} \operatorname{Hom}(Q^{\text{ét}}, K_0) = D_{\text{st}}(Q^{\text{ét}}) \subset D_{\text{st}}(Q)$.

Here $N_Q : Q^{\text{mult},C} \to \operatorname{Hom}(Q^{\text{ét}}, \mathbb{Z})$ is the map induced by the monodromy pairing of Q (2.2.3).

Proof. Since both N_0 and N_Q are additive in Q and are trivial when Q is a 1-motif, it is actually enough to show that they are equal when Q is the log 1-motif $(0, \mathbb{Z}, \mathbb{Z}, 0, 0, \tau_\pi)$, where $\tau_\pi : \mathbb{Z} \times \mathbb{Z} \to K^\times$ is the pairing such that $\tau(1,1) = \pi$. In this case, $D_{\text{st}}(Q)$ is simply the weakly admissible filtered (φ, N)-module attached to a Tate curve with parameter π, and here the result is well-known; cf. [Ber04, §II.4]. ☐

Remark 2.4.13. If Q is a positively polarized log 1-motif, then, by (2.3.4), it corresponds to a semi-stable abelian variety A over K such that $T_p(A) = T_p(Q)$ and $H^1_{\text{dR}}(A) = H^1_{\text{dR}}(Q)[p^{-1}]$. So the results above can be rephrased appropriately in terms of the étale and de Rham cohomology of A. In this case, (2.4.12.1) is due to Coleman-Iovita [CI99, §II.4].

3. At the boundary of a Chai-Faltings compactification

3.1. **Chai-Faltings local models as deformation spaces of log 1-motifs.**

3.1.1. We begin with a pair (V, ψ), where V is a $\mathbb{Q}$-vector space of dimension $2g$ and ψ is a symplectic form on V. We will fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ such that ψ restricts to an alternating form

$$\psi : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}.$$

We will call such a lattice a **polarized lattice** for V.

For any ring R, set $V_R = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$; we will also denote by ψ the induced pairing on V_R. Let $V_{\mathbb{Z}}^\vee \subset V$ be the dual lattice, so that ψ induces a perfect pairing on $V_{\mathbb{Z}} \times V_{\mathbb{Z}}^\vee$, and let $d \in \mathbb{Z}_{>0}$ be such that the order of the finite group $V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$ is d^2. We will call d the **discriminant** of the polarized lattice $V_{\mathbb{Z}}$. Then $\operatorname{GSp}(V_{\mathbb{Z}[1/d]}, \psi)$ is a reductive sub-group of $\operatorname{GL}(V_{\mathbb{Z}[1/d]})$.

Let n be an integer in $\mathbb{Z}_{>0}$ such that $(n, d) = 1$. Associated with $(V_{\mathbb{Z}/n\mathbb{Z}}, \psi)$, we have the moduli stack $\mathbf{M}_{V_{\mathbb{Z}}, n, \psi}$ over $\mathbb{Z}\left[\frac{1}{n}\right]$; for any $\mathbb{Z}\left[\frac{1}{n}\right]$-scheme S, $\mathbf{M}_{V_{\mathbb{Z}}, n, \psi}(S)$ parameterizes isomorphism classes of tuples $(A, \lambda, \nu, \alpha)$, where:

- (A, λ) is a polarized abelian scheme over S;
- $\nu : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\simeq} \mu_{n,S}$ is an isomorphism of étale sheaves of groups over S.
- $\alpha : \underline{V}_{\mathbb{Z}/n\mathbb{Z}, S} \xrightarrow{\simeq} A[n]$ is an isomorphism of étale sheaves of groups over S that carries $\nu \circ \psi$ to the Weil pairing on $A[n]$ induced by λ, and is **symplectic liftable** in the sense of [Lan08, 1.3.6.2].

In particular, the symplectic liftability condition insures that the prime-to-d part of $\ker \lambda$ is isomorphic to the prime-to-d part of the constant sheaf $\underline{V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}$. It is known that the restriction of $\mathbf{M}_{V_{\mathbb{Z}}, n, \psi}$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$ is smooth.

Remark 3.1.2. We can and will make sense of this moduli space even when $g = 0$. It will be the $\mathbb{Z}\left[\frac{1}{n}\right]$-scheme $\mathbf{M}_{0,n}$ classifying isomorphisms $\underline{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\simeq} \mu_n$. In other words, it is $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{n}\right][\zeta_n]$, where ζ_n is a primitive n^{th}-root of unity.

Definition 3.1.3. Let R be a $\mathbb{Z}\left[\frac{1}{d}\right]$-algebra. We will call a filtration $\mathrm{W}_\bullet V_R \subset V_R$ **admissible** if it is $\mathrm{GSp}(V_R, \psi)$-split (cf. [Kis10, 1.1.2]). Concretely, this means that it is of the form

$$0 = \mathrm{W}_{-3} V_R \subset \mathrm{W}_{-2} V_R \subset \mathrm{W}_{-1} V_R = (\mathrm{W}_{-2} V_R)^{\perp} \subset \mathrm{W}_0 V_R = V_R,$$

where $W_{-2}V_R \subset V_R$ is an isotropic direct summand. We will call the filtration **proper** if $\mathrm{W}_{-2} V_R \neq 0$.

Definition 3.1.4. Let $\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$ be an admissible filtration, and let $r = \mathrm{rank}_{V_{\mathbb{Z}/n\mathbb{Z}}} \mathrm{W}_2 V_{\mathbb{Z}/n\mathbb{Z}}$. A **torus argument** Φ **for** $(V_{\mathbb{Z}}, \psi, \mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}})$ (cf. [Lan08, 5.4.1]) is a tuple

$$(Y, X, \lambda^{\text{ét}}, \varphi_n^{\text{ét}}, \varphi_n^{\text{mult}}),$$

where:

(1) Y and X are free $\mathbb{Z}$-modules of rank r and $\lambda^{\text{ét}} : Y \to X$ is an injective map of groups.

(2)
$$\varphi_n^{\text{ét}} : \mathrm{gr}_0^{\mathrm{W}} V_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\simeq} Y/NY;$$
$$\varphi_n^{\text{mult}} : \mathrm{W}_2 V_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\simeq} \mathrm{Hom}(X, \mathbb{Z}/n\mathbb{Z})$$

are isomorphisms of groups such that the pairing

$$\mathrm{gr}_0^{\mathrm{W}} V_{\mathbb{Z}/n\mathbb{Z}} \times \mathrm{W}_{-2} V_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\varphi_n^{\text{ét}} \times \varphi_n^{\text{mult}}} Y/NY \times \mathrm{Hom}(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\langle \lambda^{\text{ét}}(\cdot), \cdot \rangle} \mathbb{Z}/n\mathbb{Z}$$

is equal to the perfect pairing induced from ψ.

Definition 3.1.5. An **isomorphism of torus arguments** $\Phi = (Y, X, \lambda^{\text{ét}}, \varphi_n^{\text{ét}}, \varphi_n^{\text{mult}})$ and $\Phi' = (Y', X', \lambda^{', \text{ét}}, \varphi_n^{', \text{ét}}, \varphi_n^{', \text{mult}})$ for $(V_{\mathbb{Z}}, \psi, \mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}})$ is a pair of isomorphisms $\gamma_X : X' \xrightarrow{\simeq} X$ and $\gamma_Y = Y \xrightarrow{\simeq} Y'$, such that:

- $\lambda^{', \text{ét}} = \gamma_X \lambda^{\text{ét}} \gamma_Y$;
- $\varphi_n^{', \text{ét}} = \gamma_Y \circ \varphi_n^{\text{ét}}$;
- $\varphi_n^{', \text{mult}} = \gamma_X^{\vee} \circ \varphi_n^{\text{mult}}$.

3.1.6. We will be considering tuples $(\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$, where:

- $\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$ is an admissible $\mathrm{GSp}(V_{\mathbb{Z}/n\mathbb{Z}}, \psi)$-split filtration of $V_{\mathbb{Z}/n\mathbb{Z}}$;
- A torus argument
$$\Phi = (Y, X, \lambda^{\text{ét}}, \varphi_n^{\text{ét}}, \varphi_n^{\text{mult}}),$$
for $(V_{\mathbb{Z}}, \psi, \mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}})$; and
- A symplectic splitting δ of the filtration $\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$.

Definition 3.1.7. Two tuples $(\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$ and $(\mathrm{W}'_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi', \delta')$ are **equivalent** if $\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}} = \mathrm{W}'_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$, and Φ and Φ' are isomorphic torus arguments. A **cusp label for** $(V_{\mathbb{Z}}, \psi)$ **at level** n is an equivalence class of tuples of the form $(\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$.

3.1.8. Suppose that we have a cusp label for $(V_{\mathbb{Z}}, \psi)$ with representative $(\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$. We will use this representative to construct certain spaces associated with the cusp label, and interpret their complete local rings as deformation rings for log 1-motifs. One can easily check that our constructions are independent (up to isomorphism) of the choice of representative.

Fix a prime $p > 0$ such that $(p, n) = 1$, let k be a perfect field of characteristic $p > 0$ and let $W = W(k)$ be its ring of Witt vectors. Fix a lift $\mathrm{W}_\bullet V_{\mathbb{Z}}$ of $\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$ such that the induced filtration $\mathrm{W}_\bullet V$ is a $\mathrm{GSp}(V)$-split. We begin with the moduli stack $\mathbf{M}_{\mathrm{W}_\bullet} = \mathbf{M}_{\mathrm{gr}^{\mathrm{W}}_{-1} V_{\mathbb{Z}}, n, \mathrm{gr}^{\mathrm{W}}_{-1} \psi}$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$; here $\mathrm{gr}^{\mathrm{W}}_{-1} \psi$ is the alternating pairing on $\mathrm{gr}^{\mathrm{W}}_{-1} V_{\mathbb{Z}}$ obtained from ψ. As explained in [Lan08, 5.2.7.5], the space $\mathbf{M}_{\mathrm{W}_\bullet}$ is determined up to isomorphism by $\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$, and so is independent of the lift $\mathrm{W}_\bullet V_{\mathbb{Z}}$.

Let $(\mathrm{B}, \lambda^{\text{ab}})$ be the universal polarized abelian scheme over $\mathbf{M}_{\mathrm{W}_\bullet}$. Suppose that we have an object $(B_0, \lambda_0^{\text{ab}}, \nu_0, \alpha_0)$ of $\mathbf{M}_{\mathrm{W}_\bullet}(k)$ corresponding to a map of stacks $x_0^{\text{ab}} : \mathrm{Spec}\, k \to \mathbf{M}_{\mathrm{W}_\bullet}$.

Proposition 3.1.9. *The complete local ring $R^{\mathrm{ab}}_{\Phi,x_0} = \widehat{\mathscr{O}}_{\mathbf{M}_{\mathrm{W}_\bullet},x_0^{\mathrm{ab}}}$ equipped with the polarized abelian scheme $(\mathrm{B}_{R^{\mathrm{ab}}_{\Phi,x_0}}, \lambda^{\mathrm{ab}}_{R^{\mathrm{ab}}_{\Phi,x_0}})$ induced from $(\mathrm{B},\lambda^{\mathrm{ab}})$ is the universal deformation ring (over W) for the polarized abelian variety $(B_0,\lambda_0^{\mathrm{ab}})$. More precisely, $R^{\mathrm{ab}}_{\Phi,x_0}$ pro-represents the deformation groupoid $\mathrm{Def}_{(B_0,\lambda_0^{\mathrm{ab}})}$ that assigns to every artin W-algebra C with residue field $k(C)$ the category*

$$\mathrm{Def}_{(B_0,\lambda_0^{\mathrm{ab}})}(C) = \begin{pmatrix} \text{Tuples } \big((B_C,\lambda_C^{\mathrm{ab}}), i_C\big) \text{ where } (B_C,\lambda_C^{\mathrm{ab}}) \text{ is} \\ \text{a polarized abelian scheme over } C\text{; and} \\ i_C : B_C \otimes_C k(C) \xrightarrow{\simeq} B_0 \otimes_k k(C) \text{ is an isomorphism} \\ \text{of abelian varieties over } k(C). \end{pmatrix}$$

Proof. This is well-known. □

3.1.10. Now consider the $\mathbf{M}_{\mathrm{W}_\bullet}$-scheme:

$$\ddot{\mathbf{C}}_\Phi = \underline{\mathrm{Hom}}\big(\frac{1}{n}Y,\mathrm{B}\big) \times_{\underline{\mathrm{Hom}}(Y,\mathrm{B}^\vee)} \underline{\mathrm{Hom}}\big(\frac{1}{n}X,\mathrm{B}^\vee\big).$$

This is the fiber product of the diagram:

$$\begin{tikzcd} & \underline{\mathrm{Hom}}\big(\frac{1}{n}Y,\mathrm{B}\big) \arrow[d] \\ \underline{\mathrm{Hom}}\big(\frac{1}{n}X,\mathrm{B}^\vee\big) \arrow[r] & \underline{\mathrm{Hom}}(Y,\mathrm{B}^\vee), \end{tikzcd}$$

where the vertical arrow is restriction followed by post-composition with λ^{ab}, and the horizontal arrow is pull-back along the map $Y \xrightarrow{\lambda^{\text{ét}}} X \hookrightarrow \frac{1}{n}X$. $\ddot{\mathbf{C}}_\Phi$ is a smooth, proper group scheme over $\mathbf{M}_{\mathrm{W}_\bullet}$.

It is shown in [Lan08, 6.2.3.4] that there is a natural map of group schemes over $\mathbf{M}_{\mathrm{W}_\bullet}$

$$\partial : \ddot{\mathbf{C}}_\Phi \to \underline{\mathrm{Hom}}\big(\frac{1}{n}Y/Y,\mathrm{B}[n]\big)$$
$$(c_n,c_n^\vee) \mapsto c_n^\vee\lambda^{\text{ét}} - \lambda^{\mathrm{ab}}c_n,$$

whose fibers are abelian schemes over $\mathbf{M}_{\mathrm{W}_\bullet}$. The splitting δ gives us a distinguished element $b_{\Phi,\delta}$ in the image of ∂ (cf. [Lan08, 6.2.3.1]). Let $\mathbf{C}_{\Phi,\delta}$ be the fiber of ∂ over $b_{\Phi,\delta}$: this is an abelian scheme over $\mathbf{M}_{\mathrm{W}_\bullet}$.

Over $\mathbf{C}_{\Phi,\delta}$, we have the tautological maps

$$c_{n,\Phi} : \frac{1}{n}Y \to \mathrm{B}; \quad c^\vee_{n,\Phi} : \frac{1}{n}X \to \mathrm{B}^\vee.$$

Set $c_\Phi = c_{n,\Phi}|_Y$ and $c_\Phi^\vee = c^\vee_{n,\Phi}|_X$.

Suppose that we have a map $x_0^{\mathrm{sab}} : \operatorname{Spec} k \to \mathbf{C}_{\Phi,\delta}$ corresponding to an object $(B_0,\lambda_0^{\mathrm{ab}},\alpha_0,c_{n,0},c^\vee_{n,0})$ over k, where $(B_0,\lambda_0^{\mathrm{ab}},\alpha_0)$ is an object of $\mathbf{M}_{\mathrm{W}_\bullet}(k)$ and $c_{n,0} : \frac{1}{N}Y \to B_0$ and $c^\vee_{n,0} : \frac{1}{n}X \to B_0^\vee$ are maps such that $c_0^\vee\lambda^{\text{ét}} = \lambda^{\mathrm{ab}}c_0$; here, $c_0^\vee = c^\vee_{n,0}|_X$ and $c_0 = c_{n,0}|_Y$. Consider the complete local ring $R^{\mathrm{sab}}_{\Phi,x_0} = \widehat{\mathscr{O}}_{\mathbf{C}_{\Phi,\delta},x_0^{\mathrm{sab}}}$: it is naturally an algebra over the deformation ring $R^{\mathrm{ab}}_{\Phi,x_0}$ considered in (3.1.9) above. Over it we have the pair $(c_{n,\Phi,R^{\mathrm{sab}}_{\Phi,x_0}}, c^\vee_{n,\Phi,R^{\mathrm{sab}}_{\Phi,x_0}})$ inherited from the universal pair over $\mathbf{C}_{\Phi,\delta}$.

Lemma 3.1.11. *Let C be an Artin local W-algebra with residue field $k(C)$. For any fppf sheaf of groups H over C, H_0 will denote its reduction to $k(C)$. For any prime-to-p isogeny*

$\phi : A' \to A$ of abelian schemes over C, the following square of fppf sheaves over C is cartesian

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\phi"'] & A'_0 \arrow[d, "\phi_0"] \\ A \arrow[r] & A_0 \end{tikzcd}$$

In particular, given a map $f'_0 : H \to A'_0$ of fppf sheaves of groups over C, lifting f'_0 to a map $f' : H \to A'$ is equivalent to lifting $\phi_0 f'_0 : H \to A_0$ to a map $f : H \to A$.

Proof. This is clear once we note that $\ker(A \to A_0)$ is p-power torsion (cf. [Kat81, 1.1.1]). □

Let $\mathrm{Art}_{R^{\mathrm{ab}}_{\Phi,x_0}}$ be the category of Artin local $R^{\mathrm{ab}}_{\Phi,x_0}$-algebras. Note that every ring C in $\mathrm{Art}_{R^{\mathrm{ab}}_{\Phi,x_0}}$ comes equipped with a polarized abelian scheme $(B_C, \lambda^{\mathrm{ab}}_C)$ lifting $(B_0, \lambda^{\mathrm{ab}}_0)$. Consider the following two deformation functors on $\mathrm{Art}_{R^{\mathrm{ab}}_{\Phi,x_0}}$:

$$\mathrm{Def}_{(c_{n,0},c^\vee_{n,0})}(C) = \begin{pmatrix} \text{Pairs } (c_{n,C}, c^\vee_{n,C}) \text{ of maps} \\ c_{n,C} : \frac{1}{n}Y \to B_C \;\; ; \;\; c^\vee_{n,C} : \frac{1}{n}X \to B^\vee_C \\ \text{lifting } (c_{n,0}, c^\vee_{n,0}) \end{pmatrix};$$

$$\mathrm{Def}_{c^\vee_0}(C) = \left(\text{Lifts } c^\vee_C : \tfrac{1}{n}X \to B^\vee_C \text{ of } c^\vee_0\right).$$

Proposition 3.1.12. *The two deformation functors are naturally isomorphic and are relatively pro-represented over $R^{\mathrm{ab}}_{\Phi,x_0}$ by $R^{\mathrm{sab}}_{\Phi,x_0}$.*

Proof. That $R^{\mathrm{sab}}_{\Phi,x_0}$ pro-represents $\mathrm{Def}_{(c_{n,0},c^\vee_{n,0})}$ is essentially tautological, so we only have to prove the isomorphism of the two functors. There is clearly a natural map $\mathrm{Def}_{(c_{n,0},c^\vee_{n,0})} \to \mathrm{Def}_{c^\vee_0}$ obtained by restricting any lift of $c^\vee_{n,0}$ to X. We claim that this is an isomorphism. Indeed, by (3.1.11), giving a lift of $c^\vee_{n,0}$ is equivalent to giving a lift of $c^\vee_0$ (apply the lemma to the isogeny $[n] : B^\vee_C \to B^\vee_C$). Once we have the lift $c^\vee_{n,C}$ of $c^\vee_{n,0}$, consider the map $c^\vee_{n,C}\lambda^{\text{ét}} - b_{\Phi,\delta} : \frac{1}{n}Y \to B^\vee_C$: this is a lift of $\lambda^{\mathrm{ab}}_0 c_{n,0}$. Again, from (3.1.11), this time applied to λ^{ab}_C, we obtain a lift $c_{n,C}$ of $c_{n,0}$. □

3.1.13. Set

$$\Psi_n = (c_{n,\Phi} \times c^\vee_\Phi)^* \mathcal{P}_{\mathrm{B}}^{-1}; \qquad \Psi = (c_\Phi \times c^\vee_\Phi)^* \mathcal{P}_{\mathrm{B}}^{-1}.$$

Then Ψ_n is a $\mathbb{G}_m$-bi-extension of $\frac{1}{n}Y \times X$ over $\mathbf{C}_{\Phi,\delta}$, and Ψ is a $\mathbb{G}_m$-bi-extension of $Y \times X$ over $\mathbf{C}_{\Phi,\delta}$ such that $(1 \times \lambda^{\text{ét}})^*\Psi$ is a symmetric $\mathbb{G}_m$-bi-extension of $Y \times Y$.

Set

$$\mathbf{B}_{\lambda^{\text{ét}}} = \{\text{Pairings } \langle\, ,\, \rangle : Y \times X \to \mathbb{Z} : \text{ such that } \langle y, \lambda^{\text{ét}}(y')\rangle = \langle y, \lambda^{\text{ét}}(y')\rangle, \text{ for all } y, y' \in Y\}.$$

Let $\mathbf{S}_{\lambda^{\text{ét}}} = \mathbf{B}^\vee_{\lambda^{\text{ét}}}$ be its dual abelian group. Also set

$$\mathbf{B}_\Phi = \frac{1}{n}\mathbf{B}_{\lambda^{\text{ét}}}; \qquad \mathbf{S}_\Phi = \frac{1}{n}\mathbf{S}_{\lambda^{\text{ét}}}.$$

Set

$$\ddot{\mathbf{S}}_\Phi = \frac{\frac{1}{n}Y \otimes X}{\left(y \otimes \lambda^{\text{ét}}(y') - y' \otimes \lambda^{\text{ét}}(y),\, y \in Y\right)}.$$

Then $\mathbf{S}_\Phi$ is just the maximal torsion-free quotient of $\ddot{\mathbf{S}}_\Phi$. Let $\ddot{\mathbf{S}}^{\mathrm{tor}}_\Phi$ be the maximal torsion sub-group of $\ddot{\mathbf{S}}_\Phi$.

Consider the sheaf $\ddot{\mathbf{\Xi}}_{\Phi,\delta}$ over $\mathbf{C}_{\Phi,\delta}$, whose points valued in any $\mathbf{C}_{\Phi,\delta}$-scheme S are given by:

$$\ddot{\mathbf{\Xi}}_{\Phi,\delta}(S) = \begin{pmatrix} \text{Trivializations } \tau_n : \mathbf{1}_{\frac{1}{n}Y\times X} \xrightarrow{\simeq} \Psi_{n,S} \text{ over } S \text{ of } \mathbb{G}_m\text{-bi-extensions of} \\ \frac{1}{n}Y \times X \text{ inducing a symmetric trivialization of} \\ \text{the symmetric } \mathbb{G}_m\text{-bi-extension } (1\times\lambda^{\text{ét}})^*\Psi_S \text{ of } Y\times Y. \end{pmatrix}$$

This is a torsor under the $\mathbf{C}_{\Phi,\delta}$-group $\ddot{\mathbf{E}}_\Phi$ of multiplicative type with character group $\ddot{\mathbf{S}}_\Phi$. It admits a natural surjection onto the $\mathbf{C}_{\Phi,\delta}$-group $\ddot{\mathbf{E}}^{\text{tor}}_\Phi$ (again of multiplicative type) with character group $\ddot{\mathbf{S}}^{\text{tor}}_\Phi$. The splitting δ allows us to naturally pick out a certain fiber $\mathbf{\Xi}_{\Phi,\delta}$ of this surjection; then $\mathbf{\Xi}_{\Phi,\delta}$ is a torsor under the $\mathbf{C}_{\Phi,\delta}$-torus $\mathbf{E}_\Phi$ with character group $\mathbf{S}_\Phi$. Note that $\mathbf{\Xi}_{\Phi,\delta}$ is open and closed in $\ddot{\mathbf{\Xi}}_{\Phi,\delta}$. For all this, cf. [Lan08, §6.2.3].

Suppose that we have a rational, polyhedral, non-degenerate cone $\sigma \subset \mathbf{B}_\Phi \otimes \mathbb{R}$; we can then form the torus embedding $\mathbf{E}_\Phi \hookrightarrow \mathbf{E}_\Phi(\sigma)$ (cf. [KKMSD73, §I.1]). More precisely, we can consider the monoid

$$\mathbf{S}_{\Phi,\sigma} = \sigma^\vee \cap \mathbf{S}_\Phi,$$

where

$$\sigma^\vee_\alpha = \{n \in \mathbf{S}_\Phi \otimes \mathbb{R} : \langle n, s\rangle \geq 0, \text{ for all } s \in \sigma\},$$

and we set $\mathbf{E}_\Phi(\sigma) = \underline{\mathrm{Spec}}\,\mathscr{O}_{\mathbf{C}_{\Phi,\delta}}[\mathbf{S}_{\Phi,\sigma}]$. We will consider the contraction product

$$\ddot{\mathbf{\Xi}}_{\Phi,\delta}(\sigma) = \ddot{\mathbf{\Xi}}_{\Phi,\delta} \times^{\mathbf{E}_\Phi} \mathbf{E}_\Phi(\sigma).$$

This is an fs log scheme over $\mathbf{C}_{\Phi,\delta}$ in the evident way with the log structure induced by the divisor that is the complement of $\ddot{\mathbf{\Xi}}_{\Phi,\delta}$. Over any fs log scheme (S, M_S) over $\mathbf{C}_{\Phi,\delta}$, we have the $\mathbb{G}_m^{\log}$-bi-extension $\Psi^{\log}_{n,(S,\mathrm{M}_S)}$ of $\frac{1}{n}Y \times X$ induced from the $\mathbb{G}_m$-extension $\Psi_{n,S}$. For any such (S, M_S), we have:

$$\ddot{\mathbf{\Xi}}_{\Phi,\delta}(\sigma)\big((S,\mathrm{M}_S)\big) = \begin{pmatrix} \text{Trivializations } \tau_n : \mathbf{1}_{\frac{1}{n}Y\times X} \xrightarrow{\simeq} \Psi^{\log}_{n,(S,\mathrm{M}_S)} \text{ over } (S,\mathrm{M}_S) \\ \text{inducing a symmetric trivialization of } (1\times\lambda^{\text{ét}})^*\Psi^{\log}_{(S,\mathrm{M}_S)}; \\ \text{and such that, for any geometric point } \bar{s} \to S, \\ \text{and any functional } \beta : \mathrm{M}_{S,\bar{s}}/\mathscr{O}^\times_{S,\bar{s}} \to \mathbb{N}, \\ \text{the pairing } Y\times X \xrightarrow{N_{\tau,\bar{s}}} \mathrm{M}^{\text{gp}}_{S,\bar{s}}/\mathscr{O}^\times_{S,\bar{s}} \xrightarrow{\beta} \mathbb{Z} \text{ lies in } \sigma \subset \mathbf{B}_\Phi\otimes\mathbb{R} \end{pmatrix}$$

Here, $N_{\tau,\bar{s}}$ is the monodromy pairing at $\bar{s}$ associated with τ (cf. 2.2.3).

Let $\mathbf{\Xi}_{\Phi,\delta}(\sigma)$ be the closure of $\mathbf{\Xi}_{\Phi,\delta}$ in $\ddot{\mathbf{\Xi}}_{\Phi,\delta}(\sigma)$: this is an open and closed strict log sub-scheme of $\ddot{\mathbf{\Xi}}_{\Phi,\delta}(\sigma)$. Over $\mathbf{\Xi}_{\Phi,\delta}$, we have a tautological trivialization $\tau_{n,\Phi}$ of the $\mathbb{G}_m$-bi-extension $\Psi_{\Phi,c_n,c^\vee}$ of $\frac{1}{n}Y \times X$ restricting to a symmetric trivialization of the symmetric $\mathbb{G}_m$-bi-extension $(1\times\lambda^{\text{ét}})^*\Psi_{\Phi,c,c^\vee}$ of $Y \times Y$. Over $\mathbf{\Xi}_{\Phi,\delta}(\sigma)$, this induces a trivialization $\tau_{n,\Phi,\sigma}$ of the $\mathbb{G}_m^{\log}$-bi-extension $\Psi^{\log}_{\Phi,c_n,c^\vee}$ of $\frac{1}{n}Y \times X$, which restricts to a symmetric trivialization of the symmetric $\mathbb{G}_m^{\log}$-bi-extension $(1\times\lambda^{\text{ét}})^*\Psi^{\log}_{\Phi,c,c^\vee}$ of $Y \times Y$.

The stratification of $\mathbf{E}_\Phi(\sigma)$ by the orbits of $\mathbf{E}_\Phi$ gives rise to a stratification on $\mathbf{\Xi}_{\Phi,\delta}(\sigma)$ as well. There is a unique closed stratum $\mathbf{Z}_{\Phi,\delta}(\sigma)$. Suppose now that we have a map $x_0 : \mathrm{Spec}\, k \to \mathbf{\Xi}_{\Phi,\delta}(\sigma)$ landing inside the closed stratum. Equip $\mathrm{Spec}\, k$ with the log structure induced from that of $\mathbf{\Xi}_{\Phi,\delta}(\sigma)$; then we have the tuple $(B_0, \lambda_0^{\text{ab}}, \alpha_0, c_{n,0}, c^\vee_{n,0}, \tau_{n,0})$ over $\mathrm{Spec}\, k$ obtained by pull-back from the tautological tuple over $\mathbf{\Xi}_{\Phi,\delta}(\sigma)$. Let τ_0 be the trivialization of the $\mathbb{G}_m^{\log}$-bi-extension Ψ_{x_0} of $Y \times X$ induced from $\tau_{n,0}$. The tuple $(B_0, Y, X, \lambda_0^{\text{ab}}, \lambda^{\text{ét}}, c_0, c_0^\vee, \tau_0)$ then gives us a polarized log 1-motif (Q_0, λ_0) over k. Let R_{Φ,σ,x_0} be the complete local ring of $\mathbf{\Xi}_{\Phi,\delta}(\sigma)$ at x_0.

Lemma 3.1.14. *Suppose that we have an Artin local fs log W-algebra (C, M_C), and a $\mathbb{G}_m^{\log}$-torsor $E_C^{\log}$ equipped with a trivialization β_0 of the induced $\mathbb{G}_m^{\log}$-torsor $E^{\log}_{k(C)}$ over $k(C)$. Then*

the map $\beta_C \mapsto \beta_C^{\otimes n}$ sets up a bijection:

$$\begin{pmatrix}\textit{Trivializations } \beta_C \textit{ of } E_C^{\log} \\ \textit{reducing to } \beta_0\end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix}\textit{Trivializations } \beta'_C \textit{ of } \mathbb{G}_m^{\log}\textit{-torsor } \big(E_C^{\log}\big)^{\otimes n} \\ \textit{reducing to the trivialization } \beta_0^{\otimes n} \textit{ of } \big(E_{k(C)}^{\log}\big)^{\otimes n}\end{pmatrix}$$

Proof. For any β'_C on the right hand side, its 'n^{th}-roots' in $E_C^{\log}$ form a μ_n-torsor over (C, M_C). So we reduce to:

Claim 3.1.15. Suppose that we have a μ_n-torsor η_C over (C, M_C). Then there is a natural bijection:

$$\big(\text{Trivializations of } \eta_C\big) \xrightarrow{\simeq} \big(\text{Trivializations of } \eta_{k(C)}\big)$$

This last claim in turn follows from the following facts:

- The map $H^1\big(C_{\mathrm{fl}}^{\log}, \mu_n\big) \to H^1\big(k(C)_{\mathrm{fl}}^{\log}, \mu_n\big)$ is a bijection.
- The map $\mu_n(C) \to \mu_n(k(C))$ is a bijection.

The first fact is a consequence of the analogous fact for classical étale μ_n-torsors and (2.1.4)(3). The second follows because $(n,p) = 1$ and because $\ker(C^\times \to k(C)^\times)$ is a p-primary group. $\square$

Let $R^{\mathrm{sab}}_{\Phi,x_0}$ be as in (3.1.12), and consider the category of fs log Artin local $R^{\mathrm{sab}}_{\Phi,x_0}$-algebras: This will be the category $\mathrm{Art}^{\log}_{R^{\mathrm{sab}}_{\Phi,x_0}}$ whose objects are pairs $\big((C, \mathrm{M}_C), j_C\big)$, where (C, M_C) is an Artin local algebra equipped with an fs log structure, and $j_C : k \to k(C)$ is a continuous map of fs log algebras extending the natural map of underlying rings. Here, we equip $k(C)$ with the log structure induced from that of C. Note that for every such object, we have the $\mathbb{G}_m$-bi-extensions $\Psi_{n,C}$ of $\frac{1}{n}Y \times X$ and Ψ_C of $Y \times X$, and also the associated $\mathbb{G}_m^{\log}$-bi-extensions $\Psi_{n,C}^{\log}$ and $\Psi_C^{\log}$. Furthermore, via the map j_C, we can view $\tau_{n,0}$ (resp. τ_0) as a trivialization of $\Psi_{n,k(C)}^{\log}$ (resp. $\Psi_{k(C)}^{\log}$).

The category $\mathrm{Art}_W^{\log}$ is defined analogously.

Consider two deformation functors on $\mathrm{Art}^{\log}_{R^{\mathrm{sab}}_{\Phi,x_0}}$:

$$\mathrm{Def}_{\tau_{n,0}}\big((C, \mathrm{M}_C), j_C\big) = \begin{pmatrix}\text{Trivializations } \tau_{n,C} \text{ of } \Psi_{n,C} \text{ lifting } \tau_{n,0} \text{ and} \\ \text{inducing symmetric trivializations of } (1 \times \lambda^{\text{ét}})^*\Psi_C\end{pmatrix}.$$

$$\mathrm{Def}_{\tau_0}\big((C, \mathrm{M}_C), j_C\big) = \begin{pmatrix}\text{Trivializations } \tau_C \text{ of } \Psi_C \text{ lifting } \tau_0 \text{ and} \\ \text{inducing symmetric trivializations of } (1 \times \lambda^{\text{ét}})^*\Psi_C\end{pmatrix}.$$

Proposition 3.1.16.

(1) *The two deformation functors above are naturally isomorphic and are relatively pro-represented over $R^{\mathrm{sab}}_{\Phi,x_0}$ by R_{Φ,σ,x_0}.*

(2) *R_{Φ,σ,x_0} pro-represents the deformation groupoid $\mathrm{Def}_{(Q_0,\lambda_0)}$ over $\mathrm{Art}_W^{\log}$ parameterizing deformations of the polarized log 1-motif (Q_0, λ_0).*

Proof. It is clear that R_{Φ,σ,x_0} relatively pro-represents $\mathrm{Def}_{\tau_{n,0}}$. We need to show that the two deformation functors are isomorphic. It is, however, immediate from (3.1.14), that, for any $\big((C, \mathrm{M}_C), j_C\big)$ in $\mathrm{Art}^{\log}_{R^{\mathrm{sab}}_{\Phi,x_0}}$, the natural map from $\mathrm{Def}_{\tau_{n,0}}\big((C, \mathrm{M}_C), j_C\big)$ to $\mathrm{Def}_{\tau_0}\big((C, \mathrm{M}_C), j_C\big)$ is a bijection.

For the second assertion, we simply have to put the first assertion together with (3.1.9) and (3.1.12), and note that deforming (Q_0, λ_0) is equivalent to deforming the tuple $(B_0, \lambda_0^{\mathrm{ab}}, c_0, c_0^\vee, \tau_0)$. $\square$

3.2. **Explicit co-ordinates for Chai-Faltings local models.** Our goal in this section is to use the deformation theoretic description of R_{Φ,σ,x_0} to write down explicit co-ordinates for it, under the assumption that $(p,nd)=1$. Let $(B_0,Y,X,\lambda_0^{\mathrm{ab}},\lambda^{\text{ét}},c_0,c_0^\vee,\tau_0)$ be the tuple corresponding to the polarized log 1-motif (Q_0,λ_0). We will suppress the sub-scripts from now on and refer to the rings $R^{\mathrm{ab}}_{\Phi,x_0}$ from (3.1.9), $R^{\mathrm{sab}}_{\Phi,x_0}$ from (3.1.12) and R_{Φ,σ,x_0} from (3.1.16) simply as R^{ab}, R^{sab} and R, respectively.

3.2.1. Set $k=k(x_0)$: this is a finite field of characteristic p. Set $\mathbf{P}_{\Phi,\sigma}=\mathbf{S}_{\Phi,\sigma}/\mathbf{S}^\times_{\Phi,\sigma}$: this is a sharp, fs monoid. By construction, $k(x_0)$ with its induced log structure is isomorphic to $k_{\mathbf{P}_{\Phi,\sigma}}$. If there is no likelihood of confusion, we will refer to this log ring simply as k; if we need to emphasize the log structure, we will write $k_{\Phi,\sigma}$ instead.

Let $W=W(k)$, and let $W_{\mathbf{P}_{\Phi,\sigma}}$ be the log scheme associated with the pre-log structure

$$\mathrm{M}_{k_{\Phi,\sigma}}\to k\xrightarrow{[\cdot]}W,$$

where the first map is the log structure on k and the second map is the Teichmüller lift. Again, if there is no danger of confusion, we will refer to this log ring simply as W.

In particular, the construction gives us a natural splitting

$$\mathrm{M}_{W_{\mathbf{P}_{\Phi,\sigma}}}=\mathrm{M}_{k_{\Phi,\sigma}}\oplus(1+pW).$$

We can use this splitting to define a Frobenius lift on $W_{\Phi,\sigma}:=W_{\mathbf{P}_{\Phi,\sigma}}$: it will be the p-power map on $\mathrm{M}_{k_{\Phi,\sigma}}$ and the usual Frobenius automorphism on $1+pW$. This means that any log F-crystal over k can be evaluated on $W_{\Phi,\sigma}$ to give a φ-module over W.

Let $\mathbb{D}(Q_0)$ be the log F-crystal over k associated with Q_0: by the functoriality of the Dieudonné functor, it has a weight filtration

$$0=W_{-1}\mathbb{D}(Q_0)\subset W_0\mathbb{D}(Q_0)\subset W_1\mathbb{D}(Q_0)\subset W_2\mathbb{D}(Q_0)=\mathbb{D}(Q_0),$$

induced by the weight filtration on Q_0. It satisfies $W_i\mathbb{D}(Q_0)=\mathbb{D}(Q_0/W_{-i+1}Q_0)$. Note that $W_{-2}Q_0=\mathrm{Hom}(X,\mathbb{G}_m^{\log})$ and that $\mathrm{gr}^W_0 Q_0=Y[1]$ (where X,Y are as in the cusp label Φ above): this gives us natural identifications of F-crystals $W_0\mathbb{D}(Q_0)=\mathrm{Hom}(Y,\mathbf{1})$ and $\mathrm{gr}^W_2\mathbb{D}(Q_0)=X\otimes\mathbf{1}(1)$.

The polarization λ_0 gives us a perfect pairing $\psi_0:\mathbb{D}(Q_0)\times\mathbb{D}(Q_0)\to\mathbf{1}(1)$. For every i, this induces a perfect pairing $\mathrm{gr}^W_i\mathbb{D}(Q_0)\times\mathrm{gr}^W_{2-i}\mathbb{D}(Q_0)\to\mathbf{1}(1)$. In particular, the perfect pairing on $W_0\mathbb{D}(Q_0)\times\mathrm{gr}^W_2\mathbb{D}(Q_0)$ is given by the formula:

$$W_0\mathbb{D}(Q_0)\times\mathrm{gr}^W_2\mathbb{D}(Q_0)=\mathrm{Hom}(Y,\mathbf{1})\times(X\otimes\mathbf{1}(1))\to\mathbf{1}(1)\tag{3.2.1.1}$$

$$\big(f,(\lambda^{\text{ét}}\otimes 1)(y\otimes 1)\big)\mapsto f(y).\tag{3.2.1.2}$$

Note that $\lambda^{\text{ét}}\otimes 1:Y\otimes\mathbf{1}(1)\to X\otimes\mathbf{1}(1)$ is an isomorphism.

Evaluating the polarized log F-crystal $(\mathbb{D}(Q_0),\psi_0)$ on $W_{\Phi,\sigma}$ gives us a φ-module M_0 equipped with a weight filtration $W_\bullet M_0$ and a symplectic pairing $\psi_0:M_0\otimes M_0\to W(1)$ of φ-modules.

By construction, the weight filtration is $\mathrm{GSp}(M_0,\psi_0)$-split: that is, W_0M_0 is isotropic for ψ_0 and W_1M_0 is its annihilator. Let $P_{\mathrm{wt}}\subset\mathrm{GSp}(M_0,\psi_0)$ be the parabolic sub-group stabilizing $W_\bullet M_0$; let U_{wt} be its unipotent radical, and let U^{-2}_{wt} be the center of U_{wt}. It is easy to see that U^{-2}_{wt} is simply the largest sub-group of U_{wt} that acts trivially on $W_1M_0\oplus(M_0/W_0M_0)$. In particular, it is commutative, every $N\in\mathrm{Lie}\,U^{-2}_{\mathrm{wt}}$ satisfies $N^2=0$ in $\mathrm{End}(M_0)$, and we have an isomorphism of group schemes

$$\begin{aligned}\underline{\mathrm{Lie}}\,U^{-2}_{\mathrm{wt}}&\xrightarrow{\simeq}U^{-2}_{\mathrm{wt}}\\ N&\mapsto 1+N.\end{aligned}$$

The natural identifications $\mathrm{Hom}(Y,W) = W_0M_0$ and $X \otimes W(1) = \mathrm{gr}_2^W M_0$ combined with the description of the pairing in (3.2.1.1) give us further canonical identifications:

$$\text{(3.2.1.3)} \qquad \mathrm{Lie}\, U_{\mathrm{wt}}^{-2} = \left\{ \begin{array}{c} \text{Pairings } N : Y \times X \to W \text{ such that} \\ N(y, \lambda^{\text{ét}}(y')) = N(y', \lambda^{\text{ét}}(y)), \text{ for all } y, y' \in Y \end{array} \right\} = \mathbf{B}_\Phi \otimes_{\mathbb{Z}} W.$$

3.2.2. We will now construct explicit models for R^{ab} and R^{sab}, following [Fal99, §7] and [Moo98, §4]. Let $\mu_0 : \mathbb{G}_m \otimes k \to \mathrm{GSp}(M_0, \psi_0) \otimes k$ be a co-character splitting the Hodge filtration $\mathrm{Fil}^1(M_0 \otimes k)$ of $M_0 \otimes k$. Concretely, this means that we are choosing a Lagrangian decomposition

$$M_0 \otimes k = \mathrm{Fil}^1(M_0 \otimes k) \oplus (M_0 \otimes k)'.$$

By [DOR10, 4.2.17], we can actually choose a splitting co-character μ_0 that factors through $P_{\mathrm{wt}} \otimes k$. We can lift this co-character to a co-character $\mu : \mathbb{G}_m \to P_{\mathrm{wt}}$ giving a Lagrangian decomposition

$$M_0 = \mathrm{Fil}^1 M_0 \oplus M_0'$$

lifting the decomposition for $M_0 \otimes k$.

Let $U^{\mathrm{op}} \subset \mathrm{GSp}(M_0, \psi_0)$ be the opposite unipotent sub-group associated with μ. Just as for U_{wt}^{-2}, every section $N \in \underline{\mathrm{Lie}}\, U^{\mathrm{op}}$ satisfies $N^2 = 0$ in $\underline{\mathrm{End}}(M_0)$, and so U^{op} is canonically isomorphic to $\underline{\mathrm{Lie}}\, U^{\mathrm{op}}$ as a group scheme over W. Since the co-character actually factors through P_{wt}, for every i, we get a decomposition

$$W_iM_0 = (W_iM_0 \cap \mathrm{Fil}^1 M_0) \oplus (W_iM_0 \cap M_0').$$

This shows that $W_0M_0 \subset M_0' \subset W_1M_0$ and in particular implies that $U_{\mathrm{wt}}^{-2} \subset U^{\mathrm{op}}$.

Let $U^{\mathrm{sab}} = U^{\mathrm{op}}/U_{\mathrm{wt}}^{-2}$, and let $U^{\mathrm{ab}} = U^{\mathrm{op}}/\big(U^{\mathrm{op}} \cap U_{\mathrm{wt}}\big)$. U^{sab} (resp. U^{ab}) acts faithfully on $M_0^{\mathrm{sab}} = M_0/W_0M_0$ (resp. $M_0^{\mathrm{ab}} = \mathrm{gr}_1^W M_0$). Note that we have a canonical identifications $M_0^{\mathrm{sab}} = \mathbb{D}(J_0)(W)$ and $M_0^{\mathrm{ab}} = \mathbb{D}(B_0)(W)$ of φ-modules over W. Here, J_0 denotes the 1-motif $W_{-1}Q_0$; we can also think of it as the semi-abelian extension of B_0 classified by $c_0^\vee$.

In what follows, $\Box$ can be read as either sab or ab. Let $\widehat{U}^\Box$ be the completion of $U^\Box$ along the identity section, and let $\mathrm{A}^\Box$ be the formally smooth W-algebra such that $\mathrm{Spf}\,\mathrm{A}^\Box = \widehat{U}^\Box$. The identity section gives us an augmentation ideal $I^\Box \subset R^\Box$ such that $R^\Box/I^\Box = W$. Fix a basis $\{e_i\}$ for $\mathrm{Lie}\, U^\Box$, and if (x_i) are the corresponding co-ordinates on $U^\Box$, let $\varphi : \mathrm{A}^\Box \to \mathrm{A}^\Box$ be the Frobenius lift carrying x_i to x_i^p.

Let $g^\Box \in U^\Box(\mathrm{A}^\Box)$ be the universal element of $\widehat{U}^\Box$. Consider the 3-tuple $\underline{M}^\Box = (M^\Box, \varphi^\Box, \mathrm{Fil}^1 M^\Box)$, where:

$$M^\Box = M_0^\Box \otimes_W \mathrm{A}^\Box;$$

$$\varphi^\Box : \varphi^* M^\Box = \varphi^* M_0^\Box \otimes_W \mathrm{A}^\Box \xrightarrow{\varphi_0 \otimes 1} M_0^\Box \otimes_W \mathrm{A}^\Box = M^\Box \xrightarrow{g^\Box} M^\Box;$$

$$\mathrm{Fil}^1 M^\Box = \mathrm{Fil}^1 M_0^\Box \otimes_W \mathrm{A}^\Box.$$

By [Moo98, 4.4], $M^\Box$ can be endowed with a unique, topologically quasi-nilpotent connection $\nabla^\Box$, for which $\varphi^\Box$ is parallel, giving us an object $(\underline{M}^\Box, \nabla^\Box)$ in the category $\mathsf{MF}^\nabla_{[0,1]}(\mathrm{A}^\Box)$ considered in *loc. cit.*. Moreover, this latter category is equivalent to the category of p-divisible groups over $\mathrm{A}^\Box$. So we obtain p-divisible groups $\mathscr{G}^{\mathrm{sab}}$ (resp. $\mathscr{G}^{\mathrm{ab}}$) over $\mathrm{A}^{\mathrm{sab}}$ (resp. A^{ab}) deforming $J_0[p^\infty]$ (resp. $B_0[p^\infty]$). By Serre-Tate theory for 1-motifs [MS11, 1.1.3.1], this gives us a deformation J^{sab} (resp. B^{ab}) of J_0 (resp. B_0) over $\mathrm{A}^{\mathrm{sab}}$ (resp. A^{ab}). Strictly speaking, the cited result only gives us formal deformations, but the polarization λ_0^{ab} on B_0 lifts to B^{ab}, and allows us to algebraize both it and J^{sab}.

By construction, $\underline{M}^{\mathrm{sab}}$ has a weight filtration

$$0 = W_0\underline{M}^{\mathrm{sab}} \subset W_1\underline{M}^{\mathrm{sab}} \subset W_2\underline{M}^{\mathrm{sab}} = \underline{M}^{\mathrm{sab}}$$

in $\mathsf{MF}^{\nabla}_{[0,1]}(\mathrm{A}^{\square})$. This filtration arises from the corresponding weight filtration on M_0^{sab}. In particular, the polarization ψ_0 of M_0 induces a polarization ψ^{ab} of the first graded component $\mathrm{gr}^1 \underline{M}^{\mathrm{sab}} = \underline{M}^{\mathrm{ab}} \otimes_{R^{\mathrm{ab}}} R^{\mathrm{sab}}$.

Since R^{ab} is the universal deformation ring for $(B_0, \lambda_0^{\mathrm{ab}})$, there exists a unique continuous map $f^{\mathrm{ab}} : R^{\mathrm{ab}} \to \mathrm{A}^{\mathrm{ab}}$ inducing $(B^{\mathrm{ab}}, \lambda^{\mathrm{ab}})$. Similarly, the map f^{ab} extends, by the universal property of R^{sab} to a map $f^{\mathrm{sab}} : R^{\mathrm{sab}} \to \mathrm{A}^{\mathrm{sab}}$ inducing J^{sab}.

The following result can be shown as in [Moo98, 4.5] via a simple Kodaira-Spencer calculation.

Proposition 3.2.3. *f^{ab} and f^{sab} are isomorphisms.*

□

3.2.4. Thanks to (3.2.3) above, we now have a rather explicit description of both R^{sab} and the Dieudonné F-crystal attached to the universal deformation of $(J_0, \lambda_0^{\mathrm{ab}})$ over R^{sab}. We will now build upon this to get a similarly explicit description for the full log deformation ring R, and the polarized log Dieudonné F-crystal attached to the universal deformation of (Q_0, λ_0) over R. As expected, we will exhibit R as a completed toric embedding over R^{sab}.

Let $\mathbf{E}_\Phi$ be the torus over W with character group $\mathbf{S}_\Phi$, and let $\mathbf{E}_\Phi \to \mathbf{E}_\Phi(\sigma)$ be the torus embedding corresponding to $\sigma \subset \mathbf{B}_\Phi \otimes \mathbb{R}$. Fix a k-valued point β in the closed orbit of $\mathbf{E}_\Phi(\sigma)$: we will think of it as a map of monoids $\beta : \mathbf{S}_{\Phi,\sigma} \to k$ satisfying $\beta^{-1}(k^\times) = \mathbf{S}_{\Phi,\sigma}^\times$. Let $R_{\Phi,\sigma}^\beta$ be the complete local ring of $\mathbf{E}_{\Phi,\sigma}$ at β.

The natural map of monoids $\mathbf{S}_{\Phi,\sigma} \to R_{\Phi,\sigma}^\beta$ endows $R_{\Phi,\sigma}^\beta$ with the structure of a log smooth log W-algebra, and we have a surjection $R_{\Phi,\sigma}^\beta \twoheadrightarrow k_{\Phi,\sigma}$ of log W-algebras. In particular, we have a map of groups $\tau_{\beta,n} : \mathbf{S}_\Phi \to \mathrm{M}^{\mathrm{gp}}_{R^\beta_{\Phi,\sigma}}$, which we can restrict to $\mathbf{S}_{\lambda^{\text{ét}}} \subset \mathbf{S}_\Phi$ to get a map $\tau_\beta : \mathbf{S}_{\lambda^{\text{ét}}} \to \mathrm{M}^{\mathrm{gp}}_{R^\beta_{\Phi,\sigma}}$. Let $\tau_{\beta,n,0} : \mathbf{S}_\Phi \to \mathrm{M}^{\mathrm{gp}}_{k_{\Phi,\sigma}}$ and $\tau_{\beta,0} : \mathbf{S}_{\lambda^{\text{ét}}} \to \mathrm{M}^{\mathrm{gp}}_{k_{\Phi,\sigma}}$ be the induced maps. We will think of $\tau_{\beta,n}$ (resp. τ_β) as a trivialization of the trivial $\mathbb{G}_m^{\log}$-bi-extension $\mathbf{1}^{\log}_{\frac{1}{n}Y\times X}$ (resp. $\mathbf{1}^{\log}_{Y\times X}$) that induces symmetric trivializations of $\mathbf{1}^{\log}_{Y\times Y}$ when pulled back along $1 \times \lambda^{\text{ét}}$.

Let us return to the category $\mathrm{Art}_W^{\log}$ of pairs $\big((C, \mathrm{M}_C), j_C\big)$. We can use $j_C : k \to k(C)$ to think of $\tau_{\beta,n,0}$ as a trivialization of $\mathbf{1}^{\log}_{\frac{1}{n}Y\times X}$ over $k(C)$, and similarly for $\tau_{\beta,0}$.

Consider the deformation functors

$$\mathrm{Def}_{\tau_{\beta,n,0}}\big((C,\mathrm{M}_C), j_C\big) = \begin{pmatrix}\text{Trivializations } \tau_{\beta,n,C} \text{ of } \mathbf{1}_{\frac{1}{n}Y\times X} \text{ over } C \text{ lifting } \tau_{\beta,n,0}\\ \text{and inducing symmetric trivializations of } \mathbf{1}_{Y\times Y}\end{pmatrix};$$

$$\mathrm{Def}_{\tau_{\beta,0}}\big((C,\mathrm{M}_C), j_C\big) = \begin{pmatrix}\text{Trivializations } \tau_{\beta,C} \text{ of } \mathbf{1}_{Y\times X} \text{ over } C \text{ lifting } \tau_{\beta,0}\\ \text{and inducing symmetric trivializations of } \mathbf{1}_{Y\times Y}\end{pmatrix}.$$

Proposition 3.2.5. *The two deformation functors above are isomorphic and $R_{\Phi,\sigma}^\beta$ pro-represents them over W.*

Proof. This is a special case of (3.1.16)(1). □

3.2.6. Recall from (3.1.13) that we have canonical $\mathbb{G}_m$-bi-extensions Ψ_n of $\frac{1}{n}Y\times X$ and Ψ of $Y\times X$ over R^{sab}. Set $\tau_{n,0}^\beta = \tau_{n,0}\tau_{\beta,n,0}^{-1}$ and $\tau_0^\beta = \tau_0\tau_{\beta,0}^{-1}$. By construction, $\tau_{n,0}^\beta$ is a trivialization of the $\mathbb{G}_m$-bi-extension $\Psi_{n,0}$ and τ_0^β is a trivialization of Ψ_0. The tuple $(B_0, Y, X, c_0, c_0^\vee, \lambda_0^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau_0^\beta)$ corresponds to a polarized 1-motif $(Q_0^\beta, \lambda_0^\beta)$ over k.

Proposition 3.2.7. *Given a co-character $w : \mathbb{G}_m \to P_{\mathrm{wt}}$ splitting the weight filtration $W_\bullet M_0$ and commuting with the Hodge co-character $\mu : \mathbb{G}_m \to P_{\mathrm{wt}}$, we can find a canonical trivialization $\tau_{w,n}^\beta$ of Ψ_n over R^{sab} lifting $\tau_{n,0}^\beta$ and inducing a symmetric trivialization of $(1\times\lambda^{\text{ét}})^*\Psi$.*

Proof. Note that such a co-character always exists; indeed, finding one is equivalent to finding a Levi sub-group $L_{\mathrm{wt}} \subset P_{\mathrm{wt}}$ containing $\mu(\mathbb{G}_m)$.

Fix a co-character w as above. From (3.1.14), it follows that it is enough to find a trivialization τ_w^β of Ψ lifting τ_0^β and inducing a symmetric trivialization of $(1 \times \lambda^{\text{ét}})^*\Psi$. This is of course equivalent to deforming the polarized 1-motif $(Q_0^\beta, \lambda_0^\beta)$ over R^{sab}.

We first observe that there is a natural identification of polarized φ-modules $\mathbb{D}(Q_0^\beta)(W) = M_0$. Now $\operatorname{Lie} U^{\mathrm{op}}$, being the -1 eigenspace for μ, is stable under the action of $w(\mathbb{G}_m)$. $\operatorname{Lie} U_{\mathrm{wt}}^{-2}$ is by definition the -2 eigenspace for w, so we get a splitting

$$\operatorname{Lie} U^{\mathrm{op}} = \operatorname{Lie} U_{\mathrm{wt}}^{-2} \oplus \operatorname{Lie} U^{\mathrm{sab}},$$

where $\operatorname{Lie} U^{\mathrm{sab}}$ maps isomorphically onto the sum of the 0 and -1 eigenspaces for w within $\operatorname{Lie} U^{\mathrm{op}}$. Since U^{op} is isomorphic to $\underline{\operatorname{Lie}}\ U^{\mathrm{op}}$, this gives us a splitting of group schemes $U^{\mathrm{op}} = U_{\mathrm{wt}}^{-2} \times U^{\mathrm{sab}}$.

In particular, we can now view every section of U^{sab} as an automorphism of M_0.

We will use this to define an object $\underline{M}^{\mathrm{cl}}$ in $\mathsf{MF}_{[0,1]} R^{\mathrm{sab}} = \mathsf{MF}_{[0,1]} \mathrm{A}^{\mathrm{sab}}$. This will be a tuple $(M^{\mathrm{cl}}, \varphi^{\mathrm{cl}}, \operatorname{Fil}^1 M^{\mathrm{cl}})$, where:

$$M^{\mathrm{cl}} = M_0 \otimes_W \mathrm{A}^{\mathrm{sab}}; \qquad \operatorname{Fil}^1 M^{\mathrm{cl}} = \operatorname{Fil}^1 M_0 \otimes_W \mathrm{A}^{\mathrm{sab}};$$

$$\varphi^{\mathrm{cl}} : \varphi^* M^{\mathrm{cl}} = \varphi^* M_0 \otimes_W \mathrm{A}^{\mathrm{sab}} \xrightarrow{\varphi_0 \otimes 1} M_0 \otimes_W \mathrm{A}^{\mathrm{sab}} = M^{\mathrm{cl}} \xrightarrow{g^{\mathrm{sab}}} M^{\mathrm{cl}}.$$

Here, $g^{\mathrm{sab}} \in \hat{U}^{\mathrm{sab}}(\mathrm{A}^{\mathrm{sab}}) = \hat{U}^{\mathrm{sab}}(R^{\mathrm{sab}})$ is the tautological element, viewed as an automorphism of $M^{\mathrm{cl}} = M_0 \otimes_R \mathrm{A}^{\mathrm{sab}}$. We can endow $\underline{M}^{\mathrm{cl}}$ with the constant polarization $\psi_0 \otimes 1$. It evidently has a weight filtration $W_\bullet \underline{M}^{\mathrm{cl}}$ with $\underline{M}^{\mathrm{cl}}/W_0\underline{M}^{\mathrm{cl}} = \underline{M}^{\mathrm{sab}}$.

By [MS11, 1.1.3.1] again, we now obtain the polarized 1-motif $(\mathrm{Q}_w^\beta, \lambda_w^\beta)$ over R^{sab} reducing to $(Q_0^\beta, \lambda_0^\beta)$ over k. □

Fix a co-character w as above. Consider the deformation functors $\operatorname{Def}_{\tau_{n,0}}$ from (3.1.16) and $\operatorname{Def}_{\tau_{\beta,n,0}}$ from (3.2.5). We can consider both as functors on $\operatorname{Art}^{\log}_{R^{\mathrm{sab}}}$.

Proposition 3.2.8. *The functor* $\operatorname{Def}_{\tau_{n,0}} \to \operatorname{Def}_{\tau_{\beta,n,0}}$ *carrying* $\tau_{n,C}$ *to* $(\tau_{w,n}^\beta)^{-1}\tau_{n,C}$ *is an isomorphism of deformation functors over* $\operatorname{Art}^{\log}_{R^{\mathrm{sab}}}$*. In particular, there is an isomorphism* $R \xrightarrow{\simeq} R^{\mathrm{sab}} \hat{\otimes} R^\beta_{\Phi,\sigma}$ *of fs log* R^{sab}*-algebras. Under this isomorphism, the universal* τ_n *over* R *is mapped to* $\tau_{w,n}^\beta \tau_{\beta,n}$*.*

Proof. This is clear. □

3.2.9. For future reference, we summarize all the choices made in the process of constructing the above explicit model for R:

- In (3.2.2), we chose a co-character $\mu_0 : \mathbb{G}_m \otimes k \to P_{\mathrm{wt}} \otimes k$ splitting the Hodge filtration, and a lift $\mu : \mathbb{G}_m \to P_{\mathrm{wt}}$ of μ_0.
- In (3.2.4), we chose a k-valued point in the closed orbit of $\mathbf{E}_\Phi(\sigma)$. Equivalently, we chose a map of monoids $\beta : \mathbf{S}_{\Phi,\sigma} \to k$ such that $\beta^{-1}(k^\times) = \mathbf{S}^\times_{\Phi,\sigma}$.
- Finally, in (3.2.6), we chose a co-character $w : \mathbb{G}_m \to P_{\mathrm{wt}}$ splitting the weight filtration $W_\bullet M_0$ and commuting with our choice of μ. Equivalently, we chose a Levi sub-group $L_{\mathrm{wt}} \subset P_{\mathrm{wt}}$ containing the image of μ.

3.2.10. Fix some choices as in (3.2.9), and use them to give explicit co-ordinates for R as in (3.2.8). We can now give an explicit description of the polarized log F-crystal $\mathbb{D}(\mathrm{Q})$ over R that is attached to the universal deformation (Q, λ) of (Q_0, λ_0) over R.

Since R is log smooth over W, giving a log F-crystal over R amounts to giving a triple (M, φ_M, ∇_M), where (M, φ_M) is a finite free φ-module over R, and ∇_M is a topologically

quasi-nilpotent logarithmic connection on M for which φ_M is parallel. Giving a Dieudonné log F-crystal over R amounts to giving a tuple $(M, \varphi_M, \mathrm{Fil}^1 M, \nabla_M)$, where (M, φ_M, ∇_M) corresponds to a log F-crystal over R and $\mathrm{Fil}^1 M \subset M$ is a direct such that $(M, \varphi_M, \mathrm{Fil}^1 M)$ is an object in $\mathrm{BT}^{\varphi}_{/R}$.

We take $M = M_0 \otimes_W R$ and $\mathrm{Fil}^1 M = \mathrm{Fil}^1 M_0 \otimes_W R$. Equip M with the constant polarization $\psi_M = \psi_0 \otimes 1$. Define φ_M as follows: write M as $M^{\mathrm{cl}} \otimes_{R^{\mathrm{sab}}} R$, where M^{cl} is as in the proof of (3.2.7), and set $\varphi_M = \varphi^{\mathrm{cl}} \otimes 1$.

Note that M^{cl} is equipped with a natural connection ∇^{cl}, for which φ^{cl} is parallel; for example, this follows from the fact that M^{cl} is the evaluation of the Dieudonné crytal of Q^{β}_{w} at R^{sab}. To finish we will describe ∇_M in terms of its connection matrix $\theta \in \mathrm{End}(M_0) \otimes_W \hat{\Omega}^{1,\log}_{R/W}$. In fact, θ will lie in $\mathrm{Lie}\, U^{\mathrm{op}} \otimes_W \hat{\Omega}^{1,\log}_{R/W}$: its component in $\mathrm{Lie}\, U^{\mathrm{sab}} \otimes_W \hat{\Omega}^{1,\log}_{R/W}$ will be the image of the connection matrix of ∇^{cl}, and its component in $\mathrm{Lie}\, U^{-2}_{\mathrm{wt}} \otimes_W \hat{\Omega}^{1,\log}_{R/W} = \mathbf{B}_{\Phi} \otimes_W \hat{\Omega}^{1,\log}_{R/W}$ will correspond to the natural map $\mathbf{S}_{\Phi} \xrightarrow{\mathrm{dlog}} \hat{\Omega}^{1,\log}_{R^{\beta}_{\Phi,\sigma}/W} \to \hat{\Omega}^{1,\log}_{R/W}$. Here, we are using the splitting $\mathrm{Lie}\, U^{\mathrm{op}} = \mathrm{Lie}\, U^{\mathrm{sab}} \oplus \mathrm{Lie}\, U^{-2}_{\mathrm{wt}}$ afforded by the choice of w above.

From the description of Q in (3.2.8) in terms of Q^{β}_{w} (equivalently $\tau_{w,n}$) and $\tau_{\beta,n}$, we obtain the following:

Proposition 3.2.11. *The tuple $(M, \varphi_M, \mathrm{Fil}^1 M, \nabla_M, \psi_M)$ is the one associated with the polarized log F-crystal $\mathbb{D}(\mathrm{Q})$ over R by the correspondence described above.*

□

3.2.12. Let R^{an} be the global ring of functions over the rigid analytic space $\widehat{U}^{\mathrm{an}}$ attached to the formal scheme $\widehat{U}$, and let $R^{\mathrm{an}}[\ell_a : a \in \mathrm{M}_R]$ be the R-algebra freely generated by the variables ℓ_a indexed by M_R. We set

$$R^{\mathrm{an},\log} = \frac{R^{\mathrm{an}}[\ell_a : a \in \mathrm{M}_R]}{\left(\begin{array}{c} \ell_{ab} - \ell_a - \ell_b, \text{ for all } a, b \in \mathrm{M}_R; \\ \ell_r - \log(r) : \text{ for all } r \in R^{\mathrm{an}} \text{ such that } |1 - r(x)|, \text{ for all } x \in \widehat{U}(\mathscr{O}_{\overline{K}_0}) \end{array} \right)}.$$

The ring $R^{\mathrm{an},\log}$ is endowed with a natural logarithmic connection

$$\begin{aligned} \nabla : R^{\mathrm{an},\log} &\to R^{\mathrm{an},\log} \otimes_R \hat{\Omega}^{1,\log}_{R/W} \\ \ell(a) &\mapsto \mathrm{dlog}(a). \end{aligned}$$

We can also equip it with a natural continuous extension of the endomorphism φ of R that carries ℓ_a to $p\ell_a$, for all $a \in \mathrm{M}^{\mathrm{gp}}_R$. We will extend the augmentation map $R \to W$ to the unique map $R^{\mathrm{an},\log} \to W$ carrying ℓ_a to 0, for all $a \in \mathbf{S}_{\Phi} \setminus \mathbf{S}^{\times}_{\Phi,\sigma}$. Set $M^{\mathrm{an},\log} = M \otimes_R R^{\mathrm{an},\log}$, so that we have a φ-equivariant identification $M^{\mathrm{an},\log} \otimes_{R^{\mathrm{an},\log}} W = M_0 \left[\frac{1}{p}\right]$. Equip $M^{\mathrm{an},\log}$ with the diagonal logarithmic connection.

Proposition 3.2.13. *There exists a unique φ-equivariant, parallel section*

$$\xi : M_0^{\otimes} \left[\frac{1}{p}\right] \to M^{\mathrm{an},\log,\otimes}.$$

Proof. This is essentially [Vol03, Theorem 9]. However, for later use, we will need an explicit version of this isomorphism, which we now present. First, let $R^{\mathrm{sab},\mathrm{an}}$ be the global ring of functions over the analytic space $\widehat{U}^{\mathrm{sab},\mathrm{an}}$, and let M^{cl} be as in (3.2.7). With this, we can associate the $R^{\mathrm{sab},\mathrm{an}}$-module $M^{\mathrm{cl},\mathrm{an}} = M^{\mathrm{cl}} \otimes_{R^{\mathrm{sab}}} R^{\mathrm{sab},\mathrm{an}}$. We will first define the φ-equivariant

(and necessarily parallel) section

$$\begin{aligned}\xi^{\mathrm{cl}} : M_0 &\to M^{\mathrm{cl,an}}\\ m &\mapsto \lim_n \varphi^n(m).\end{aligned}$$

This is simply Dwork's trick. Note that its definition is clearly compatible with tensor operations on both sides.

Next, we consider the map

$$\mathbf{S}_\Phi \to \mathrm{M}_R^{\mathrm{gp}} \xrightarrow{a\mapsto \ell_a} R^{\mathrm{an,log}}.$$

This can be viewed as an element

$$A^{\log} \in \mathbf{B}_\Phi \otimes R^{\mathrm{an,log}} = \mathrm{Lie}\, U_{\mathrm{wt}}^{-2} \otimes_W R^{\mathrm{an,log}} \subset \mathrm{End}(M^{\mathrm{an,log}}).$$

Set $\xi^{\log} = \exp(A^{\log}) \in U_{\mathrm{wt}}^{-2}(R^{\mathrm{an,log}})$. One can now easily check that $\xi^{\log}$ commutes with ξ^{cl} and that $\xi = \xi^{\mathrm{cl}}\xi^{\log}$ is the unique section whose existence had to be shown. □

3.2.14. An important consequence of (3.2.13), exhibited in [Vol03, Theorem 6], is the following: Let L/K_0 be a finite extension, let $L_0 \subset L$ be the maximal unramified sub-extension, and let $\pi \in \mathscr{O}_L$ be a uniformizer, allowing us to fix a branch $\log_\pi$ of the p-adic logarithm satisfying $\log_\pi(\pi) = 0$. For any map $x : R \to \mathscr{O}_L$ of log rings, the choice of logarithm gives us a unique extension $x^{\log} : R^{\mathrm{an,log}} \to L$ carrying ℓ_a to $\log_\pi(x^\sharp(a))$. Evaluating the isomorphism of (3.2.13) along this map gives us an isomorphism of L-vector spaces:

$$M_0 \otimes_W L \xrightarrow{\simeq} H^1_{\mathrm{dR}}(Q_x)\left[\frac{1}{p}\right]. \tag{3.2.14.1}$$

Let $l = \mathscr{O}_L/(\pi)$ equipped with its induced log structure, and let $x_0 : R \to l$ be the reduction of x. We can identify M_0 with $\mathbb{D}(Q_{x_0})(W(l)_{\mathbb{N}})$ as a φ-module over W, giving us:

$$\mathbb{D}(Q_{x_0})(W(l)_{\mathbb{N}}) \otimes_{W(l)} L \xrightarrow{\simeq} H^1_{\mathrm{dR}}(Q_x)\left[\frac{1}{p}\right]. \tag{3.2.14.2}$$

The following result is an immediate consequence of the log smoothness of R and the construction of the Hyodo-Kato isomorphism in (2.4.10), which is also accomplished via parallel transport.

Proposition 3.2.15. *The isomorphism in (3.2.14.2) agrees with the Hyodo-Kato isomorphism from (2.4.10.1).*

□

3.2.16. Let $\varphi_0 \otimes 1 : \varphi^* M \to M$ be the scalar extension of $\varphi_0 : \varphi^* M_0 \to M_0$. For any $h \in \mathrm{End}(M\left[\frac{1}{p}\right])$, set

$$\varphi(h) = \varphi_0(\varphi^* h)\varphi_0^{-1} \in \mathrm{End}\left(M\left[\frac{1}{p}\right]\right).$$

Let $g^{\mathrm{sab}} \in U^{\mathrm{sab}}(R)$ be the image of the universal element of $U^{\mathrm{sab}}(R^{\mathrm{sab}})$, and let $\Phi(g^{\mathrm{sab}})$ be the convergent product

$$\Phi(g^{\mathrm{sab}}) = g^{\mathrm{sab}}\varphi(g^{\mathrm{sab}})\varphi^2(g^{\mathrm{sab}})\varphi^3(g^{\mathrm{sab}})\cdots \in \mathrm{GL}(M^{\mathrm{an}}).$$

The following corollary is immediate from the description of ξ in the proof of (3.2.13).

Corollary 3.2.17. *For any φ-invariant element $s_0 \in M_0^{\otimes}\left[\frac{1}{p}\right]$, we have:*

$$\xi(s_0) = \xi^{\log}\Phi(g^{\mathrm{sab}})(s_0 \otimes 1).$$

□

3.3. Tate tensors.

Definition 3.3.1. A collection of tensors $\{s_{\alpha,0}\}_{\alpha\in I}\subset M_0^{\otimes}\left[\frac{1}{p}\right]$ is a **collection of Tate tensors for** M_0 if:

(1) Each s_α, for $\alpha\in I$, is φ-invariant.
(2) The point-wise stabilizer of the collection is a reductive sub-group $G_{K_0}\subset \mathrm{GSp}(M_0,\psi_0)_{K_0}$.

3.3.2. Let $\{s_{\alpha,0}\}_{\alpha\in I}\subset M_0^{\otimes}\left[\frac{1}{p}\right]$ be a collection of Tate tensors. Let $U^{-2}_{\mathrm{wt},G,K_0}=U^{-2}_{\mathrm{wt},K_0}\cap G_{K_0}$; then we have

$$\mathrm{Lie}\,U^{-2}_{\mathrm{wt},G,K_0}\subset \mathrm{Lie}\,U^{-2}_{\mathrm{wt},K_0}=\mathbf{B}_\Phi\otimes K_0.$$

Set $\mathbf{B}_{\Phi_G}=\mathbf{B}_\Phi\cap\mathrm{Lie}\,U^{-2}_{\mathrm{wt},G,K_0}$: this is a direct summand of $\mathbf{B}_\Phi$. Set $\mathbf{S}_{\Phi_G}=\mathbf{B}^\vee_{\Phi_G}$: this is a quotient of $\mathbf{S}_\Phi$, and is again a free abelian group. Let $\mathbf{E}_{\Phi_G}$ be the torus over W with character group $\mathbf{S}_{\Phi_G}$; it is a sub-torus of $\mathbf{E}_\Phi$. Let $\sigma_G=\sigma\cap\left(\mathbf{B}_{\Phi_G}\otimes\mathbb{R}\right)$, and let $\mathbf{S}_{\Phi_G,\sigma_G}\subset\mathbf{S}_{\Phi_G}$ be the fs monoid associated with the non-degenerate, rational, polyhedral cone σ_G. Attached to this, we have the torus embedding

$$\mathbf{E}_{\Phi_G}\hookrightarrow\mathbf{E}_{\Phi_G}(\sigma_G).$$

Finally, set $\mathbf{P}_{\Phi_G,\sigma_G}=\mathbf{S}_{\Phi_G,\sigma_G}/\mathbf{S}^\times_{\Phi_G,\sigma_G}$.

Definition 3.3.3. A collection of Tate tensors $\{s_{\alpha,0}\}\subset M_0^{\otimes}\left[\frac{1}{p}\right]$ satisfies the **continuity property** if the natural map of monoids $\mathbf{S}_{\Phi,\sigma}\to\mathbf{S}_{\Phi_G,\sigma_G}$ is **continuous**; that is, if only invertible elements in $\mathbf{S}_{\Phi,\sigma}$ are mapped to invertible elements in $\mathbf{S}_{\Phi_G,\sigma_G}$.

If $\{s_{\alpha,0}\}$ satisfies the continuity property, in making our choices as in (3.2.9), we can find $\beta:\mathbf{S}_{\Phi,\sigma}\to k$ such that β factors through $\mathbf{S}_{\Phi_G,\sigma_G}$. Let $R^\beta_{\Phi_G,\sigma_G}$ be the complete local ring of $\mathbf{E}_{\Phi_G}(\sigma_G)$ at β: this is the normalization of a quotient ring of $R^\beta_{\Phi,\sigma}$; cf. [Har89, 3.1]. *We will assume that we have made such a choice of β for the rest of this section.*

3.3.4. Here is one way to obtain Tate tensors for M_0 satisfying the continuity property: Let $\mathscr{O}_K$ be the ring of integers in a finite extension $K/\mathbb{Q}_p$ with residue field k and maximal ideal $\mathfrak{m}_K\subset\mathscr{O}_K$. Equip it with its canonical log structure. Suppose that we have:

(1) A polarized log 1-motif (Q_x,λ_x) over $\mathscr{O}_K$;
(2) A continuous map of log rings $j_x:k_{\Phi,\sigma}\to\mathscr{O}_K/\mathfrak{m}_K$, where we have equipped the right hand side with the log structure induced from $\mathscr{O}_K$; and
(3) An identification $j_x^*(Q_0,\lambda_0)=(Q_{x,0},\lambda_{x,0})\coloneqq(Q_x,\lambda_x)\otimes_{\mathscr{O}_K}\mathscr{O}_K/\mathfrak{m}_K$.

In the language of §3.1.13, we have an object of $\mathrm{Def}_{(Q_0,\lambda_0)}(\mathscr{O}_K)$; this is of course equivalent to giving a local map $x:R\to\mathscr{O}_K$ of fs log algebras.

Let $\Lambda_x=T_p(Q_x)$, and suppose, in addition, that we have a collection $\{s_{\alpha,x,\text{ét}}\}\subset\Lambda_x^{\otimes}\left[\frac{1}{p}\right]$ of Galois-invariant tensors over L_x defining a reductive sub-group $G_{\mathbb{Q}_p}\subset\mathrm{GSp}\big(\Lambda_x,\psi_x\big)_{\mathbb{Q}_p}$, where ψ_x is the $\mathbb{Z}_p(1)$-valued symplectic form on Λ_x induced from λ_x.

Via the p-adic comparison isomorphism (2.4.10.2), we now obtain φ-invariant tensors $\{s_{\alpha,x,\mathrm{st}}\}\subset\mathbb{D}(Q_{x,0})(W_{\mathbb{N}})^{\otimes}\left[\frac{1}{p}\right]$ satisfying $N(s_{\alpha,x,\mathrm{st}})=0$, for all α. Note that the condition $N(s_{\alpha,x,\mathrm{st}})=0$ ensures that the φ-invariance of s_α is *independent* of the choice of Frobenius lift on $W_{\mathbb{N}}$. Choose any Frobenius lift on $W_{\mathbb{N}}$: this amounts to a choice of a splitting $\mathrm{M}_{W_{\mathbb{N}}}=\mathrm{M}_{\mathscr{O}_K/\mathfrak{m}_K}\oplus(1+pW)$. Since we have already chosen a splitting $\mathrm{M}_{W_{\Phi,\sigma}}=\mathrm{M}_{k_{\Phi,\sigma}}\oplus(1+pW)$, there now exists a unique map $\widetilde{j}_x:W_{\Phi,\sigma}\to W_{\mathbb{N}}$ lifting j_x and respecting the chosen splittings. In particular, $\widetilde{j}_x$ is φ-equivariant, and so we have an equality of φ-modules $M_0=\mathbb{D}(Q_{x,0})(W_{\mathbb{N}})$, giving us φ-invariant tensors $\{s_{\alpha,0}\}=\{s_{\alpha,x,\mathrm{st}}\}\subset M_0^{\otimes}\left[\frac{1}{p}\right]$.

We have the following:

Lemma 3.3.5. $\{s_{\alpha,0}\}$ *is a collection of Tate tensors satisfying the continuity property.*

□

3.3.6. Fix a collection of Tate tensors $\{s_{\alpha,0}\} \subset M_0^{\otimes}\left[\frac{1}{p}\right]$ satisfying the continuity property. Let $R^{\mathrm{an,log}}$ and $M^{\mathrm{an,log}}$ be as in (3.2.12). The Tate tensors $\{s_{\alpha,0}\}$ give rise to a parallel φ-invariant tensors

$$\{s_\alpha\} = \{\xi(s_{\alpha,0} \otimes 1)\} \subset M^{\mathrm{an,log},\otimes},$$

where ξ is as in (3.2.13).

Suppose that L/K_0 is a finite extension and that $x : R \to \mathscr{O}_L$ is a local map of log W-algebras. Then, for each choice of uniformizer $\pi \in \mathscr{O}_L$, the Hyodo-Kato isomorphism (2.4.10.1) carries the Tate tensors $\{s_{\alpha,0}\}$ to tensors $\{s_{\alpha,\pi,x}\} \subset H^1_{\mathrm{dR}}(Q_x)\left[\frac{1}{p}\right]^{\otimes}$. According to (3.2.15) these tensors are exactly the evaluation of $\{s_\alpha\}$ under the map $x^{\log} : R^{\mathrm{an,log}} \to L$ induced by the branch of logarithm $\log_\pi$ attached to π. Given a different choice of uniformizer $\pi' \in \mathscr{O}_L$, we have:

$$s_{\alpha,\pi',x} = \exp(\log(\pi'\pi^{-1})N_x)(s_{\alpha,\pi,x}),$$

where N_x is the monodromy map on $M_x \coloneqq H^1_{\mathrm{dR}}(Q_x)\left[\frac{1}{p}\right]$. This follows, for example, from the explicit description of ξ in (3.2.13).

Lemma 3.3.7. *Suppose that* $s_{\alpha,\pi,x}$ *belongs to* $\mathrm{Fil}^0 M_x^{\otimes}$*; then it is invariant under monodromy, and is therefore determined independently of the choice of* π*.*

Proof. Let $L_0 \subset L$ be the maximal unramified sub-extension. Set $M_{x,0} = M_0 \otimes_W L_0$; then $M_{x,0}$ is equipped with the isomorphism

$$\xi_{x,\pi} : M_{x,0} \otimes_{L_0} L \xrightarrow{\simeq} M_x.$$

Along this isomorphism, the monodromy N_x descends to a map $N_{x,0}$ on $M_{x,0}$, and this gives $M_{x,0}$ the structure of a weakly admissible filtered (φ, N)-module with Hodge-Tate weights in $\{0,1\}$. If $s_{\alpha,\pi,x} = \xi_{x,\pi}(s_{\alpha,0})$ belongs to $\mathrm{Fil}^0 M_x$, then, since $N_{x,0}\varphi = p\varphi N_{x,0}$, $N_{x,0}(s_{\alpha,0}) = N_x(s_{\alpha,\pi,x})$ belongs to the intersection $(M_{x,0}^{\otimes})^{\varphi=p^{-1}} \cap \mathrm{Fil}^0 M_x^{\otimes}$.

But, if we 'forget' the monodromy $N_{x,0}$, the remaining filtered φ-module is still weakly admissible. Therefore, one easily sees that the intersection in question must be zero: each element in $(M_{x,0}^{\otimes})^{\varphi=p^{-1}}$ spans a φ-stable sub-space of slope -1, and so must have a Hodge polygon with negative slopes. In particular, $s_{\alpha,\pi,x}$ must be invariant under monodromy. □

Definition 3.3.8. We will say that the collection $\{s_\alpha\}$ is **Hodge at** x if, for some (hence any) choice of uniformizer π, we have $\{s_{\alpha,\pi,x}\} \subset \mathrm{Fil}^0 M_x^{\otimes}$. In this case, by (3.3.7), the specializations $\{s_{\alpha,\pi,x}\}$ are determined independently of π, and so we will write them simply as $\{s_{\alpha,x}\}$.

Definition 3.3.9. For any finite extension L/K_0, and any quotient T of $R_{\mathscr{O}_L}$, write $\mathrm{LM}(T)$ for the set of continuous maps of log W-algebras $x : R \to \mathscr{O}_{\overline{K}_0}$ that factor through T.

Note that a continuous map $x : R \to \mathscr{O}_{\overline{K}_0}$ is a map of log algebras precisely when the associated map of monoids $x^\sharp : \mathbf{S}_{\Phi,\sigma} \to \mathscr{O}_{\overline{K}_0}$ takes only non-zero values.

Lemma 3.3.10. *Fix a finite extension* $L/\mathbb{Q}_p$ *and* $x : R \to \mathscr{O}_L$ *in* $\mathrm{LM}(R)$*. For any* $h \in U^{-2}_{\mathrm{wt}}(L)$*, set* $\mathrm{Fil}^1_h M_x = h \cdot \mathrm{Fil}^1 M_x$*. Then the tuple* $(M_{x,0}, \mathrm{Fil}^1_h M_x, \varphi, N_{x,0})$ *is still weakly admissible.*

Proof. This follows because h acts trivially on $W_1 M_{x,0}$, as well as on $\mathrm{gr}^W_2 M_{x,0}$, and because the category of weakly admissible filtered (φ, N)-modules is closed under extensions within the category of filtered (φ, N)-modules. □

Lemma 3.3.11. *Let $L/\mathbb{Q}_p$ be a finite extension, and let $(D, \mathrm{Fil}^\bullet D_L, \varphi, N)$ be a weakly admissible filtered (φ, N)-module over L. Suppose that $H \subset \mathrm{GL}(D)$ is a reductive sub-group that is the point-wise stabilizer of a collection of tensors*

$$\{v_\beta\} \subset (D^\otimes)^{\varphi=1,N=0} \bigcap \mathrm{Fil}^0 D_L^\otimes.$$

Then $\mathrm{Fil}^\bullet D_L$ is split by a co-character $\mu : \mathbb{G}_{m,L} \to H_L$.

Proof. This follows from the argument in [Kis10, 1.4.5]. □

Proposition 3.3.12. *Let $\widehat{\mathbf{E}}_\Phi$ be the completion of $\mathbf{E}_{\Phi,W}$ along the identity section. Fix $x : R \to \mathscr{O}_L$ in $\mathrm{LM}(R)$. For $\alpha \in I$, set $s_\alpha^{\mathrm{cl}} = \xi^{\mathrm{cl}}(s_{\alpha,0} \otimes 1)$, and let $s_{\alpha,x}^{\mathrm{cl}}$ denote its specialization at x. Then the following are equivalent:*

1. *(1) $\{s_\alpha\}$ is Hodge at x.*
2. *(2) There exists $u \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L)$ such that $\{(1 - \log(u))s_{\alpha,x}^{\mathrm{cl}}\} \subset \mathrm{Fil}^0 M_x^\otimes$, and, for any such u, $ux^\sharp : \mathbf{S}_\Phi \to L^\times$ factors through $\mathbf{S}_{\Phi_G}$.*

Proof. We first note that, given $u \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L) = \mathrm{Hom}(\mathbf{S}_\Phi, 1 + \mathfrak{m}_L)$, $\log(u) \in \mathbf{B}_\Phi \otimes L = \mathrm{Lie}\, U_{\mathrm{wt},L}^{-2}$ is the element attached to the composition:

$$\mathbf{S}_\Phi \xrightarrow{u} 1 + \mathfrak{m}_L \xrightarrow{\log} L.$$

Now, fix a uniformizer π in L. Let $\xi_{x,\pi}^{\log} \in U_{\mathrm{wt}}^{-2}(L) = 1 + \mathrm{Lie}\, U_{\mathrm{wt},L}^{-2}$ be the automorphism obtained by viewing the composition

$$\mathbf{S}_\Phi \xrightarrow{x^\sharp} L^\times \xrightarrow{\log_\pi} L$$

as an element of $\mathbf{B}_\Phi \otimes L = \mathrm{Lie}\, U_{\mathrm{wt},L}^{-2}$. Then $\{s_\alpha\}$ is Hodge at x if and only if $\{\xi_{x,\pi}^{\log}(s_{\alpha,x}^{\mathrm{cl}})\} \subset \mathrm{Fil}^0 M_x^\otimes$.

Suppose now that (2) holds. Then, for each α,

$$\xi_{x,\pi}^{\log}(s_{\alpha,x}^{\mathrm{cl}}) = (1 - \log(u))(s_{\alpha,x}^{\mathrm{cl}}) \in \mathrm{Fil}^0 M_x^\otimes.$$

This shows that (2)⇒(1).

For the other implication, assume that $\{s_\alpha\}$ is Hodge at x. Then it follows from the proof of (3.3.7) that the monodromy N_x satisfies $N_x(s_{\alpha,0}) = 0$, for all α. This means that the composition

$$\mathbf{S}_\Phi \xrightarrow{x^\sharp} L^\times \to L^\times/\mathscr{O}_L^\times$$

factors through $\mathbf{S}_{\Phi_G}$. Since $\beta : \mathbf{S}_{\Phi,\sigma} \to k$ was chosen to factor through $\mathbf{S}_{\Phi_G,\sigma_G}$, we can find $u \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L)$ such that $ux^\sharp$ factors through $\mathbf{S}_{\Phi_G}$. So we find:

$$(1 - \log(u))(s_{\alpha,x}^{\mathrm{cl}}) = \xi_{x,\pi}^{\log}(s_{\alpha,x}^{\mathrm{cl}}) = s_{\alpha,x} \in \mathrm{Fil}^0 M_x^\otimes.$$

To finish the proof, suppose that $v \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L)$ is another element such that $(1-\log(v))(s_{\alpha,x}^{\mathrm{cl}})$ lies in $\mathrm{Fil}^0 M_x^\otimes$ for all α. Then, for all α,

$$\xi_{x,\pi}^{\log}(1 + \log(v))\big((1 - \log(v))(s_{\alpha,x}^{\mathrm{cl}})\big) = s_{\alpha,x} \in \mathrm{Fil}^0 M_x^\otimes. \tag{3.3.12.1}$$

Let $G_{x,v} \subset \mathrm{GL}(M_x)$ be the point-wise stabilizer of the collection $\{(1-\log(v))(s_{\alpha,x}^{\mathrm{cl}})\}$. Then, by (3.3.10) and (3.3.11), the filtration $\mathrm{Fil}^1 M_x \subset M_x$ is split by a co-character $\mu : \mathbb{G}_{m,L} \to G_{x,v}$.

Consider the map

$$\begin{aligned} \mathrm{End}(M_x) &\to \oplus_{\alpha\in I} M_x^\otimes \\ f &\mapsto (f((1 - \log(v))(s_{\alpha,x}^{\mathrm{cl}})))_{\alpha\in I}. \end{aligned}$$

It is easy to see that this map is $G_{x,v}$-equivariant, and, using the fact that $\mathrm{Fil}^1 M_x$ is $G_{x,v}$-split, we see that the pre-image of $\oplus_{\alpha\in I} \mathrm{Fil}^0 M_x^\otimes$ under this map is precisely $\mathrm{Lie}\, G_{x,v} + \mathrm{Lie}\, P_L$.

Here, $P \subset \mathrm{GL}(M_0)$ is the parabolic sub-group stabilizing the Hodge filtration on M_0 and $P_L \subset \mathrm{GL}(M_x) = \mathrm{GL}(M_0) \otimes_W L$ is the stabilizer of $\mathrm{Fil}^1 M_x$.

Moreover, we have

(3.3.12.2)

$$\operatorname{Lie} U^{-2}_{\mathrm{wt},L} \cap \left(\operatorname{Lie} G_{x,v} + \operatorname{Lie} P_L\right) = \operatorname{Lie} U^{-2}_{\mathrm{wt},L} \cap \operatorname{Lie} G_{x,v} = \operatorname{Lie} U^{-2}_{\mathrm{wt},L} \cap \operatorname{Lie} G_L = \operatorname{Lie} U^{-2}_{\mathrm{wt},G,L}.$$

The second to last equality can be deduced as follows: By its definition and by (3.2.17), we have:

$$G_{x,v} = (1 - \log(v))\Phi(g^{\mathrm{sab}})_x G_L \Phi(g^{\mathrm{sab}})_x^{-1}(1 + \log(v)).$$

Given this, we only have to observe that $U^{-2}_{\mathrm{wt},L}$ commutes with both $(1 - \log(v))$ and $\Phi(g^{\mathrm{sab}})_x$ (for the latter, note that U^{-2}_{wt} is invariant under conjugation by φ and that it commutes with g^{sab}).

From (3.3.12.1) and (3.3.12.2), we find that $\xi^{\log}_{x,\pi}(1+\log(v))$ belongs to $U^{-2}_{\mathrm{wt},G,L}$. In particular, since we already know that $\xi^{\log}_{x,\pi}(1+\log(u))$ belongs to $U^{-2}_{\mathrm{wt},G,L}$, this shows that $\log(vu^{-1})$ belongs to $\operatorname{Lie} U^{-2}_{\mathrm{wt},G,L}$, implying in turn that vu^{-1} factors through $\mathbf{S}_{\Phi_G}$. Therefore, $vx^\sharp = (vu^{-1})ux^\sharp$ must also factor through $\mathbf{S}_{\Phi_G}$. □

3.3.13. Let Gr (resp. $\mathrm{Gr}^{\mathrm{sab}}$) be the Grassmannian over W that parameterizes direct summands $\mathrm{Fil}^1 M_0 \subset M_0$ (resp. $\mathrm{Fil}^1 M_0^{\mathrm{sab}} \subset M_0^{\mathrm{sab}}$) of rank $\operatorname{rank} \mathrm{Fil}^1 M_0$. Let $\overline{P}_{\mathrm{wt},G,K_0}$ be the image of P_{wt,G,K_0} in $\mathrm{GL}(M_0^{\mathrm{sab}})\left[\frac{1}{p}\right]$. Let $\mathrm{Gr}_G \subset \mathrm{Gr}\left[\frac{1}{p}\right]$ (resp. $\mathrm{Gr}^{\mathrm{sab}}_G \subset \mathrm{Gr}^{\mathrm{sab}}\left[\frac{1}{p}\right]$) be the sub-scheme consisting of G_{K_0}-split filtrations $\mathrm{Fil}^1 M_0\left[\frac{1}{p}\right] \subset M_0\left[\frac{1}{p}\right]$ (resp. $\overline{P}_{\mathrm{wt},G,K_0}$-split filtrations $\mathrm{Fil}^1 M_0^{\mathrm{sab}}\left[\frac{1}{p}\right] \subset M_0^{\mathrm{sab}}\left[\frac{1}{p}\right]$). Here, given a closed sub-group $H \subset \mathrm{GL}(V)$, we say that a filtration $\mathrm{Fil}^\bullet V$ is H-split if it is split by a co-character $\mu : \mathbb{G}_m \to H$.

Lemma 3.3.14. *Let* $\mathrm{U} \subset \mathrm{Gr}_G$ *be the open sub-scheme consisting of summands* $\mathrm{Fil}^1 M_0$ *such that* $\mathrm{Fil}^1 M_0 \cap W_0 M_0 = 0$*. Then the map carrying* $\mathrm{Fil}^1 M_0$ *to its image in* M_0^{sab} *induces a flat map* $\mathrm{U} \to \mathrm{Gr}^{\mathrm{sab}}_G$ *whose fibers have constant dimension* $\dim U^{-2}_{\mathrm{wt},G,K_0}$*. In particular, for any connected component* $Z \subset \mathrm{Gr}^{\mathrm{sab}}_G$*, we have:*

$$\dim Z = d - \dim U^{-2}_{\mathrm{wt},G,K_0}.$$

Here, $d = \dim G_{K_0} - \dim P_{G,y}$*, where* $y \in \mathrm{U}(\overline{K}_0)$ *is any point mapping into* Z *and* $P_{G,y} \subset \mathrm{GL}(M_0) \otimes_W \overline{K}_0$ *is the parabolic sub-group stabilizing the attached filtration* $\mathrm{Fil}^1_y M_0 \subset M_0$*.*

Proof. According to [DOR10, 4.2.17], for any point $y \in \mathrm{Gr}_G(\overline{K}_0)$, the attached filtration $\mathrm{Fil}^1_y(M_0 \otimes \overline{K}_0)$ can be split by a co-character $\mu : \mathbb{G}_m \to P_{\mathrm{wt},G,\overline{K}_0}$. This shows that we have a map $\mathrm{U} \to \mathrm{Gr}^{\mathrm{sab}}_G$. We claim that all the fibers of this map have dimension $\dim U^{-2}_{\mathrm{wt},G,K_0}$. This will finish the proof, since d is simply the dimension of any connected component of U lying above Z.

To prove our claim, we simply have to note that the natural action of $U^{-2}_{\mathrm{wt},G,K_0}$ on U makes U a $U^{-2}_{\mathrm{wt},G,K_0}$-torsor over $\mathrm{Gr}^{\mathrm{sab}}_G$. □

3.3.15. Suppose that L/K_0 is a finite extension and that $x : R \to \mathscr{O}_L$ in $\mathrm{LM}(R)$ is such that $\{s_\alpha\}$ is Hodge at x. Let $G_x \subset \mathrm{GL}(M_x)$ be the point-wise stabilizer of $\{s_{\alpha,x}\}$. Set $P_{G,x} = G_x \cap P_L$, and set $d_x = \dim G_{K_0} - \dim P_{G,x}$. It follows from (3.3.11) that $\mathrm{Fil}^1 M_x$ is G_x-split, so that $P_{G,x}$ is a maximal parabolic sub-group of G_x.

Lemma 3.3.16. *Suppose that* T *is a quotient domain of* $R_{\mathscr{O}_L}$ *such that* $\mathrm{LM}(T)$ *is non-empty, and such that, for every* $x \in \mathrm{LM}(T)$, $\{s_\alpha\}$ *is Hodge at* x*. Let* T^{sab} *be the image of* $R^{\mathrm{sab}}_{\mathscr{O}_L}$ *in* T*.*

Then

$$\dim T^{\mathrm{sab}} \leq d - \dim U^{-2}_{\mathrm{wt},G,K_0} + 1, \tag{3.3.16.1}$$

where $d = d_x$, for some (hence any) $x \in \mathrm{LM}(T)$.

Proof. The isomorphism

$$\xi^{\mathrm{cl}} : M_0^{\mathrm{sab}} \otimes_W R^{\mathrm{sab,an}} \xrightarrow{\simeq} M^{\mathrm{sab,an}}$$

over $\widehat{U}^{\mathrm{sab,an}}$ along with the filtration $(\xi^{\mathrm{cl}})^{-1}(\mathrm{Fil}^1 M^{\mathrm{sab,an}})$ defines a map of analytic spaces $f : \widehat{U}^{\mathrm{sab,an}} \to \mathrm{Gr}^{\mathrm{sab,an}}$. One can check that this map is unramified; that is, it induces injections on tangent spaces. This essentially follows from the universality of R^{sab}.

By our hypothesis, f carries $(\mathrm{Spf}\, T^{\mathrm{sab}})^{\mathrm{an}}$ into $\mathrm{Gr}^{\mathrm{sab,an}}_{G,\mathscr{O}_L}$. The result now follows from (3.3.14). □

Proposition 3.3.17. *Suppose that there exists a quotient domain T of $R_{\mathscr{O}_L}$ that enjoys the following properties:*

(1) $\mathrm{LM}(T)$ is non-empty, and for every $x \in \mathrm{LM}(T)$, $\{s_\alpha\}$ is Hodge at x.
(2) $\dim T = d + 1$, where $d = \dim G_{K_0} - \dim P_{G,x}$, for one (hence any) $x \in \mathrm{LM}(T)$.

Let T^{sab} be the image of $R^{\mathrm{sab}}_{\mathscr{O}_L}$ in T. Given $u \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L)$, write $u \cdot R^\beta_{\Phi_G,\sigma_G,\mathscr{O}_L}$ for the ring of functions on $u \cdot (\mathrm{Spf}\, R^\beta_{\Phi_G,\sigma_G,\mathscr{O}_L}) \subset \mathrm{Spf}\, R_{\Phi,\sigma,\mathscr{O}_L}$. Then:

(1) $\mathrm{rank}\, \mathbf{B}_{\Phi_G} = \dim U^{-2}_{\mathrm{wt},G,K_0}$.
(2) For any $y : T \to \mathscr{O}_L$ in $\mathrm{LM}(T)$ with restriction $y^{\mathrm{sab}} = y|_{T^{\mathrm{sab}}}$, there exists $u_y \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L)$ such that

$$T \otimes_{T^{\mathrm{sab}},y^{\mathrm{sab}}} L = u_y^{-1} \cdot R^\beta_{\Phi_G,\sigma_G,L}$$

as quotients of $R^\beta_{\Phi,\sigma,L} = R \otimes_{R^{\mathrm{sab}},y^{\mathrm{sab}}} L$.

Proof. Note that $\widehat{\mathbf{E}}_\Phi$ acts naturally on $\mathrm{Spf}\, R^\beta_{\Phi,\sigma}$ via translation. It is this translation action that appears in (2).

Fix $y : T \to \mathscr{O}_L$ and let $\mathfrak{p}_y \subset T^{\mathrm{sab}}$ be the kernel of $y^{\mathrm{sab}} = y|^{\mathrm{sab}}_T$. Let $\mathfrak{P} \subset T$ be any prime minimal over $\mathfrak{p}_y T$ such that $\mathfrak{P} \cap T^{\mathrm{sab}} = \mathfrak{p}_y$. Then, by [Mat89, Theorem 15.1], we have:

$$\dim T_{\mathfrak{P}} \leq \dim T^{\mathrm{sab}}_{\mathfrak{p}_y} \leq d - \dim U^{-2}_{\mathrm{wt},G,K_0}.$$

Here, we are using the bound (3.3.16.1) on the dimension of T^{sab}. This implies, using the fact that every complete local Noetherian ring is catenary [Mat89, Theorem 29.4], that:

$$\dim T/\mathfrak{P} = \dim T - \dim T_{\mathfrak{P}} \geq \dim U^{-2}_{\mathrm{wt},G,K_0} + 1.$$

We will now treat $T/\mathfrak{p}_y T$ as a quotient of $\mathscr{O}_L \otimes_{R^{\mathrm{sab}},y} R = R^\beta_{\Phi,\sigma,\mathscr{O}_L}$. Fix $u_y \in \widehat{\mathbf{E}}_\Phi(\mathscr{O}_L)$ such that

$$\{(1 - \log(u_y))(s^{\mathrm{cl}}_{\alpha,y})\} \subset \mathrm{Fil}^0 M_y^{\otimes}.$$

This is possible by (3.3.12). Moreover, (2) of *loc. cit.* shows that $\mathrm{LM}(T/\mathfrak{p}_y T)$ lies within $u_y^{-1} \cdot \mathrm{LM}(R^\beta_{\Phi_G,\sigma_G,\mathscr{O}_L})$ (here, we are viewing elements of $\mathrm{LM}(R^\beta_{\Phi,\sigma})$ as continuous monoid homomorphisms $\mathbf{S}_{\Phi,\sigma} \to \mathscr{O}_{\overline{K}_0} \setminus \{0\}$). In particular, we find:

$$\dim U^{-2}_{\mathrm{wt},G,K_0} \geq \dim R^\beta_{\Phi_G,\sigma_G}\left[\frac{1}{p}\right] \geq \dim(T/\mathfrak{p}_y T)\left[\frac{1}{p}\right] \geq \dim U^{-2}_{\mathrm{wt},G,K_0}.$$

This shows that $\mathrm{rank}\, \mathbf{B}_{\Phi_G} = \dim R^\beta_{\Phi_G,\sigma_G} - 1 = \dim U^{-2}_{\mathrm{wt},G,K_0}$, thus proving (1), and also proves (2) □

Definition 3.3.18. We will say that a quotient domain T that satisfies properties (1) and (2) in (3.3.17) above is **adapted to** $\{s_{\alpha,0}\}$.

3.3.19. Recall that, by construction (cf. 3.1.13), we have over R^{sab} the $\mathbb{G}_m$-bi-extension Ψ_n of $\frac{1}{n}Y \times X$. The trivializations of this $\mathbb{G}_m$-bi-extension inducing a symmetric trivialization of $(1 \times \lambda^{\text{ét}})^*\Psi$ form an $\mathbf{E}_\Phi$-torsor $\mathbf{\Xi}_{\Phi,R^{\mathrm{sab}}}$ over $\operatorname{Spec} R^{\mathrm{sab}}$. Let $\mathbf{\Xi}^G_{\Phi,R^{\mathrm{sab}}}$ be the induced $\mathbf{E}_\Phi/\mathbf{E}_{\Phi_G}$-torsor.

By definition, $\mathbf{\Xi}_{\Phi,R^{\mathrm{sab}}}$ is nothing but $\operatorname{Spec}\bigl(\bigoplus_{l\in\mathbf{S}_\Phi}\Psi_n(l)^{-1}\bigr)$. Here, if $l = \sum_i [y_i \otimes x_i] \in \mathbf{S}_\Phi$, for $y_i \in \frac{1}{n}Y$ and $x_i \in X$, we set $\Psi_n(l) = \bigotimes_i (y_i \otimes x_i)^*\Psi_n$. Similarly, if $\mathbf{S}^G_\Phi = \ker(\mathbf{S}_\Phi \to \mathbf{S}_{\Phi_G})$, then $\mathbf{\Xi}^G_{\Phi,R^{\mathrm{sab}}} = \operatorname{Spec}\bigl(\bigoplus_{l\in\mathbf{S}^G_\Phi}\Psi_n(l)^{-1}\bigr)$.

Over $\mathbf{\Xi}_{\Phi,R^{\mathrm{sab}}}$, we have the tautological trivialization τ of $\mathbf{\Xi}_{\Phi,R^{\mathrm{sab}}}$. In particular, for each $l \in \mathbf{S}^G_\Phi$, there exists a canonical trivialization $\tau(l) \in \Psi_n(l)^{-1} \otimes_{R^{\mathrm{sab}}} \mathscr{O}_{\mathbf{\Xi}_{\Phi,R^{\mathrm{sab}}}}$. Given an R^{sab}-algebra C, we will say that $\mathbf{\Xi}^G_\Phi$ is **tautologically trivialized** over C if, for any $l \in \mathbf{S}^G_\Phi$, the image of $\tau(l)$ in $\Psi_n(l)^{-1} \otimes_{R^{\mathrm{sab}}} \mathscr{O}_{\mathbf{\Xi}_{\Phi,C}}$ lies in $\Psi_n(l)^{-1} \otimes_{R^{\mathrm{sab}}} C$. In this case, the collection $\{\tau(l)\}_{l\in\mathbf{S}^G_\Phi}$ provides a trivialization of $\mathbf{\Xi}^G_{\Phi,C}$, and we can attach to it a $\mathbf{E}_{\Phi_G}$-torsor $\mathbf{\Xi}_{\Phi_G,C}$ over C such that

$$\mathbf{\Xi}_{\Phi,C} = \mathbf{\Xi}_{\Phi_G,C} \times^{\mathbf{E}_{\Phi_G}} \mathbf{E}_\Phi.$$

Proposition 3.3.20. *Let T be a quotient domain of $R_{\mathscr{O}_L}$ adapted to $\{s_{\alpha,0}\}$. Then:*

(1) $\mathbf{\Xi}^G_\Phi$ is tautologically trivialized over T^{sab}.

(2) Let $\mathbf{\Xi}_{\Phi_G,T^{\mathrm{sab}}} \hookrightarrow \mathbf{\Xi}_{\Phi_G,T^{\mathrm{sab}}}(\sigma_G)$ be the twisted torus embedding attached to the cone $\sigma_G \subset \mathbf{B}_{\Phi_G} \otimes \mathbb{R}$. Let R_G be the complete local ring of $\mathbf{\Xi}_{\Phi_G,T^{\mathrm{sab}}}(\sigma_G)$ at x_0, and let $\breve{R}_G$ be the image of R in R_G; then $\breve{R}_G = T$ as quotients of $R_{\mathscr{O}_L}$.

Proof. Over R, the tautological trivialization τ gives rise to the universal deformation τ_n of $\tau_{n,0}$. We saw in (3.1.16) that, given our choice of co-character $w : \mathbb{G}_m \to P_{\mathrm{wt},G}$ and the map $\beta : \mathbf{S}_{\Phi_G,\sigma_G} \to k$, we have $\tau_n = \tau^\beta_{w,n}\tau_{\beta,n}$, where $\tau^\beta_{w,n}$ is a trivialization of Ψ_n over R^{sab}, and $\tau_{\beta,n} : \mathbf{S}_\Phi \to \mathrm{M}^{\mathrm{gp}}_{R^\beta_{\Phi,\sigma}}$ is the natural map, viewed as a trivialization of the trivial $\mathbb{G}^{\log}_m$-bi-extension of $\frac{1}{n}Y \times X$.

Fix $y : T \to \mathscr{O}_{L'}$ in $\mathrm{LM}(T)$. By (3.3.17)(2), for all $l \in \mathbf{S}^G_\Phi$, we have

$$y^*\tau_{\beta,n}(l) = u_y(l)^{-1} \in 1 + \mathfrak{m}_{L'}.$$

So, for all $l \in \mathbf{S}^G_\Phi$,

$$y^*\tau_n(l) = u_y(l)^{-1}(y^*\tau^\beta_{w,n}(l)) \in \Psi_n(l)^{-1} \otimes_{R^{\mathrm{sab}},y^{\mathrm{sab}}} \mathscr{O}_{L'}.$$

Let $\mathfrak{p}_y \subset T^{\mathrm{sab}}$ be the kernel of $y^{\mathrm{sab}} : T^{\mathrm{sab}} \to \mathscr{O}_{L'}$. Fix a basis $\mathcal{B}$ for the free T^{sab}-algebra $\mathscr{O}_{\mathbf{\Xi}_{\Phi,T^{\mathrm{sab}}}}$ such that $1 \in \mathcal{B}$. Then for any $1 \neq e \in \mathcal{B}$, we saw above that the e-co-ordinate of $\tau_n(l)$ maps to 0 in $T^{\mathrm{sab}}/\mathfrak{p}_y$. Since this holds for all $y \in \mathrm{LM}(T)$, we see that the e-co-ordinate of $\tau_n(l)$ must in fact be 0 and that $\tau_n(l)$ must therefore lie in $\Psi_n(l)^{-1} \otimes_{R^{\mathrm{sab}}} T^{\mathrm{sab}}$. This shows (1).

By construction, $\breve{R}_G$ is a domain, and we have:

$$\dim \breve{R}_G = \dim T^{\mathrm{sab}} + \operatorname{Lie} U^{-2}_{\mathrm{wt},G,K_0} \leq d + 1 = \dim T.$$

Let us use the trivialization $\tau^\beta_{w,n}$ to identify $\mathbf{\Xi}_{\Phi,T^{\mathrm{sab}}}$ with the trivial torsor $\mathbf{E}_{\Phi,T^{\mathrm{sab}}}$. Then, for any T^{sab}-algebra C, we have:

$$\mathbf{\Xi}_{\Phi_G,T^{\mathrm{sab}}}(C) = \{f : \mathbf{S}_\Phi \to C^\times :\ f(l) = \tau_{\beta,n}(l), \text{ for all } l \in \mathbf{S}^G_\Phi\}.$$

In particular, if $y : T \to \mathscr{O}_{L'}$ is in $\mathrm{LM}(T)$, then treating L' as a T^{sab} algebra via y^{sab}, we have:

$$\mathbf{\Xi}_{\Phi_G,T^{\mathrm{sab}}}(L') = \{f : \mathbf{S}_\Phi \to (L')^\times :\ f(l) = u_y(l)^{-1}, \text{ for all } l \in \mathbf{S}^G_\Phi\}.$$

Using (3.3.12)(2), we now see that $y^\sharp : \mathbf{S}_\Phi \to (L')^\times$ is an element of $\mathbf{\Xi}_{\Phi_G,T^{\mathrm{sab}}}(L')$. This shows $\mathrm{LM}(T) \subset \mathrm{LM}(\breve{R}_G)$ and proves (2). $\square$

4. Compactifications of Shimura varieties of Hodge type

4.1. **Shimura varieties and absolute Hodge cycles.** This is essentially a resumé of the first part of [Kis10, §2], but we will be using Pink's slightly more general definition of Shimura data from [Pin90], rather than Deligne's original definition from [Del71].

Definition 4.1.1. A **Shimura datum** is a triple (G, X, h), where G is a connected reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-homogeneous space, and $h : X \to \mathrm{Hom}\big(\mathbb{S}, G_{\mathbb{R}}\big)$ is a $G(\mathbb{R})$-equivariant map (here, $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ is the Deligne torus) such that:

(1) For any $x \in X$, the composite

$$\mathbb{S} \xrightarrow{h_x} G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(G))$$

defines a Hodge structure of type $(-1, 1), (0, 0), (1, -1)$ on $\mathrm{Lie}(G)$;

(2) For any $x \in X$, $h_x(i)$ is a Cartan involution of $G_{\mathbb{R}}$;

(3) G^{ad} has no $\mathbb{Q}$-simple factors whose $\mathbb{R}$-points form a compact group.

Usually h will be clear from context, and we will use the pair (G, X) to refer to the Shimura datum.

A map $\iota : (G_1, X_1) \to (G_2, X_2)$ of Shimura data consists of a pair (ι_1, ι_2), where $\iota_1 : G_1 \to G_2$ is a map of $\mathbb{Q}$-groups, and $\iota_2 : X_1 \to X_2$ is a $G_1(\mathbb{R})$-equivariant map compatible with h_1 and h_2 in the obvious sense. It is an **embedding** if ι_1 is a closed embedding.

Definition 4.1.2. The **weight co-character** $w_0 : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is the composition $\mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S} \xrightarrow{h_x} G_{\mathbb{R}}$, for $x \in X$. Here, the first map is the natural inclusion. The definition of a Shimura datum ensures that w_0 maps into the center of $G_{\mathbb{R}}$ and is independent of the choice of x.

Assumption 4.1.3. *We will assume from now on that w_0 is defined over $\mathbb{Q}$.*

Definition 4.1.4. For any $x \in X$, let $\mu_x : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the co-character

$$\begin{array}{ccccccc} \mathbb{G}_{m,\mathbb{C}} & \to & \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} & \xrightarrow{\simeq} & \mathbb{S}_{\mathbb{C}} & \xrightarrow{h_x} & G_{\mathbb{C}} \\ z & \mapsto & (z, 1) & & & & \end{array}$$

The **reflex field** $E(G, X) \subset \mathbb{C}$ of (G, X) is the field of definition of the conjugacy class of μ_h in G. It is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$.

Let $\mathbb{A}_f$ be the ring of finite adéles, let $K \subset G(\mathbb{A}_f)$ be a compact open sub-group of the adélic points of G. We will write $K = K^p K_p$, where $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p \subset \mathbb{A}_f$ denotes the sub-ring of adéles with trivial p-component.

By results of Baily-Borel, Shimura, Deligne, Milne, Borovoi and others (see [Mil90, §4.5]), the double coset space

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

has the natural structure of an algebraic variety over $\mathbb{C}$ with a canonical model $\mathrm{Sh}_K(G, X)$ over the reflex field $E(G, X)$.

Lemma 4.1.5. *Let $\iota : (G_1, X_1) \hookrightarrow (G_2, X_2)$ be an embedding of Shimura data, let $K_{2,p} \subset G_1(\mathbb{Q}_p)$ be a compact open sub-group, and let $K_{1,p} = K_{2,p} \cap G_2(\mathbb{Q}_p)$.*

(1) For any compact open sub-group $K_1^p \subset G_1(\mathbb{A}_f^p)$, we can find a compact open sub-group $K_2^p \subset G_2(\mathbb{A}_f^p)$ containing K_1^p such that ι induces a closed embedding

$$i : \mathrm{Sh}_{K_1^p K_{1,p}}(G_1, X_1) \hookrightarrow \mathrm{Sh}_{K_2^p K_{2,p}}(G_2, X_2)$$

defined over $E(G_1, X_1)$.

(2) *For $K_1 = K_1^p K_{1,p}$ sufficiently small, we can choose $K_2 = K_2^p K_{2,p}$ such that, for any compact open sub-group $K_2' \subset K_2$, the map*

$$i' : \mathrm{Sh}_{K_1'}(G_1, X_1) \hookrightarrow \mathrm{Sh}_{K_2'}(G_2, X_2)$$

is again a closed embedding; here $K_1' = K_2' \cap G_1(\mathbb{A}_f)$.

Proof. The first assertion is [Kis10, 2.1.2].

For the second, we choose K_1 small enough so that K_2 can be chosen to be **neat** (cf. [Lan08, 1.4.1.8]). In this case, for any $K_2' \subset K_2$ and $r = 1, 2$, the map

$$\mathrm{Sh}_{K_r'}(G_r, X_r) \to \mathrm{Sh}_{K_r}(G_r, X_r)$$

is finite étale. In particular, the map i' is finite and unramified, and so, to check that it is a closed immersion, it is enough to show that it is injective on $\mathbb{C}$-valued points. Suppose that we have two points (x, g) and (y, h) in $X_1 \times G_1(\mathbb{A}_f)$ mapping to the same point in $\mathrm{Sh}_{K_2'}(G_2, X_2)$. This means that they map to the same point in $\mathrm{Sh}_{K_1}(G_1, X_1)$ as well. So we can find $\gamma_r \in G_r(\mathbb{Q})$, for $r = 1, 2$, $k_1 \in K_1$ and $k_2' \in K_2$ such that

$$(y, h) = (\gamma_1 x, \gamma_1 g k_1) = (\gamma_2 x, \gamma_2 g k_2').$$

This implies $\gamma_2^{-1}\gamma_1 \in \mathrm{Stab}_{G_1(\mathbb{Q})}(x) \bigcap g K_2' g^{-1}$. Since K_2' is neat, this last intersection is trivial, which means that $\gamma_2 = \gamma_1 \in G_1(\mathbb{Q})$ and $k_2' = k_1 \in K_1'$. □

Definition 4.1.6. Let V be a $\mathbb{Q}$-vector-space equipped with a symplectic form ψ. The **Siegel Shimura datum** associated to (V, ψ) is the pair $(\mathrm{GSp}(V, \psi), \mathrm{S}^{\pm}(V, \psi))$, where $\mathrm{S}^{\pm}(V, \psi)$ is the $\mathrm{GSp}(V, \psi)(\mathbb{R})$-conjugacy class of maps $h : \mathbb{S} \to \mathrm{GSp}(V, \psi)_{\mathbb{R}}$ such that:

(1) h induces a Hodge structure of type $(1, 0), (0, 1)$ on V, so that we have a corresponding decomposition
$$V_{\mathbb{C}} = V_h^{1,0} \oplus V_h^{0,1};$$
(2) The symmetric form $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

The reflex field of a Siegel Shimura datum is $\mathbb{Q}$.

Following [Pin90, 2.6], we can also make sense of a Siegel Shimura datum when $V = 0$. We set $\mathrm{GSp}(0) := \mathbb{G}_m$, and $\mathrm{S}^{\pm}(0)$ to be the set of square roots of -1 in $\mathbb{C}$ with the obvious action of $\mathbb{R}^{\times}$. We equip $\mathrm{S}^{\pm}(0)$ with the constant map $h : \mathrm{S}^{\pm}(0) \to \mathrm{Hom}(\mathbb{S}, \mathbb{G}_{m,\mathbb{R}})$ carrying either square root to the norm map $z \mapsto z\overline{z}$. We denote this Shimura datum by $(\mathrm{GSp}(0), \mathrm{S}^{\pm}(0))$.

4.1.7. Let $(\mathrm{GSp}, \mathrm{S}^{\pm})$ be the Siegel Shimura datum associated to (V, ψ) (we assume for now that $V \neq 0$), and let $K = K^p K_p \subset \mathrm{GSp}(\mathbb{A}_f)$ be a compact open sub-group. For K^p sufficiently small, $\mathrm{Sh}_K(\mathrm{GSp}, \mathrm{S}^{\pm})$ can be interpreted as the fine moduli space of polarized abelian varieties with level structure. To be more precise, we fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ such that ψ restricts to a bilinear form on $V_{\mathbb{Z}}$ and such that $V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$ is stable under K. Let $V_{\mathbb{Z}}^{\vee} \subset V$ be the dual lattice with respect to ψ, and let $d = \sharp\big(V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}\big)$.

For an abelian scheme A over a $\mathbb{Q}$-scheme S, and for a rational prime ℓ, let $T_{\ell}(A)$ be the Tate module of A: this is an ℓ-adic sheaf over S. Set $T_{\widehat{\mathbb{Z}}}(A) = \prod_{\ell \text{ prime}} T_{\ell}(A)$. Then, for any $\mathbb{Q}$-scheme S, $\mathrm{Sh}_K(\mathrm{GSp}, \mathrm{S}^{\pm})(S)$ parameterizees isomorphism classes of tuples (A, λ, η), where

- A is an abelian scheme over S;
- λ is a polarization of A of degree d;
- η is a section of the étale sheaf

$$\underline{\mathrm{Isom}}\big((\underline{V}_{\widehat{\mathbb{Z}},S}, [\psi]), (T_{\widehat{\mathbb{Z}}}(A), [\psi_{\lambda}])\big)/K.$$

Here, $[x]$ denotes the line spanned by x, and ψ_λ is the Weil pairing on $T_{\widehat{\mathbb{Z}}}(A)$ induced by the polarization λ. The group K acts on the sheaf of isomorphisms via pre-composition. For more details, see [Del71, §4] or [Kot92, §5]. We see in particular that, for K^p sufficiently small, there exists a universal abelian scheme $\mathcal{A}$ over $\mathrm{Sh}_K(\mathrm{GSp},\mathrm{S}^{\pm})$.

If $V=0$, then $\mathrm{Sh}_K(\mathrm{GSp}(0),\mathrm{S}^{\pm})$ is the finite étale $\mathbb{Q}$-scheme attached to the $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$-set $\mathbb{A}_f^\times/\mathbb{Q}^{>0}K$.

Definition 4.1.8. A Shimura datum (G,X) is of **Hodge type** if it admits an embedding

$$(G,X)\hookrightarrow(\mathrm{GSp}(V,\psi),\mathrm{S}^{\pm}(V,\psi))$$

into a Siegel Shimura datum.

Remark 4.1.9. Note that, unless $V=0$, any such Shimura datum will be a Shimura datum in the sense of Deligne. From now on, unless otherwise specified, we will assume $V\neq 0$.

4.1.10. Let (G,X) be a Shimura datum of Hodge type with reflex field $E=E(G,X)$ equipped with an embedding

$$(G,X)\hookrightarrow(\mathrm{GSp}(V,\psi),\mathrm{S}^{\pm}).$$

Let $K=K^pK_p\subset G(\mathbb{A}_f)$ be a compact open subgroup. By (4.1.5), we can find $K'\subset\mathrm{GSp}(\mathbb{A}_f)$ containing K such that the map $\mathrm{Sh}_K(G,X)\to\mathrm{Sh}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$ is an embedding defined over $E=E(G,X)$. Moreover, we can ensure that K^p and K'^p are sufficiently small, and fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}\subset V$ as above, so that $\mathrm{Sh}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$ admits an interpretation as a fine moduli space of polarized abelian schemes with level structure. Let $h:\mathcal{A}\to\mathrm{Sh}_K(G,X)$ be the pull-back of the universal family of abelian varieties over $\mathrm{Sh}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$.

Suppose that we have a finite collection of tensors $\{s_{\alpha,B}\}\subset V^{\otimes}$ whose pointwise stabilizer in GSp is G. Let $\boldsymbol{V}_{\mathrm{dR},E}=H^1_{\mathrm{dR}}(\mathcal{A}/\mathrm{Sh}_K(G,X))$ be the first relative de Rham cohomology of $\mathcal{A}$ over $\mathrm{Sh}_K(G,X)$: this is a vector bundle with integrable connection over $\mathrm{Sh}_K(G,X)$. From [Kis10, §2.2], we see that the tensors $\{s_{\alpha,B}\}$, via the de Rham isomorphism, give rise to parallel tensors

$$\{s_{\alpha,\mathrm{dR}}\}\subset H^0\bigl(\mathrm{Sh}_K(G,X),F^0\boldsymbol{V}^{\otimes}_{\mathrm{dR},E}\bigr)^{\nabla=0}.$$

Moreover, for any α, any field extension κ of E, any point $x\in\mathrm{Sh}_K(G,X)(\kappa)$, and any choice of algebraic closure $\overline{\kappa}$ of κ, we get a $\mathrm{Gal}(\overline{\kappa}/\kappa)$-invariant tensor $s_{\alpha,\text{ét},x}\in H^1_{\text{ét}}(\mathcal{A}_{x,\bar\kappa},\mathbb{Q}_p)^{\otimes}$. Given any choice of embeddings $\sigma:\bar\kappa\hookrightarrow\mathbb{C}$ and $\iota:\mathbb{Q}_p\hookrightarrow\mathbb{C}$, under the isomorphisms

$$H^1_{\mathrm{dR}}(\mathcal{A}_x)\otimes_{\kappa,\sigma}\mathbb{C}\xrightarrow{\simeq}H^1(\mathcal{A}_{x,\sigma}(\mathbb{C}),\mathbb{C})\xrightarrow{\simeq}H^1_{\text{ét}}(\mathcal{A}_{x,\bar\kappa,\sigma},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p,\iota}\mathbb{C},$$

$s_{\alpha,\mathrm{dR},x}$ is carried to $s_{\alpha,\text{ét},x}$. All of these results are easy consequences of the main result of [DMOS82]: 'Hodge implies absolutely Hodge for abelian varieties over $\mathbb{C}$'.

We also have one additional piece of compatibility between $s_{\alpha,\mathrm{dR},x}$ and $s_{\alpha,\text{ét},x}$. For this, consider the case where κ is a finite extension of E_v, the completion at v for some place $v|p$ of E. Then we also have the p-adic comparison isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{A}_x)\otimes_\kappa B_{\mathrm{dR}}\xrightarrow{\simeq}H^1_{\text{ét}}(\mathcal{A}_{x,\bar\kappa},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}B_{\mathrm{dR}}.$$

Proposition 4.1.11. *Under the p-adic comparison isomorphism above, $s_{\alpha,\mathrm{dR},x}$ is carried to $s_{\alpha,\text{ét},x}$.*

Proof. This is the main result of [Bla94], which applies directly when $\mathcal{A}_x$ is in fact defined over a number field. For the generality we need, as pointed out in [Moo98, 5.6.3], we can either appeal to a trick of Lieberman as in [Vas99, 5.2.16], or we can directly use the fact that $\mathcal{A}_x$ arises from the family $\mathcal{A}$ defined over the number field E. □

4.2. **Compactifications in characteristic** 0.

4.2.1. Let (G, X) be a Shimura datum, and suppose $G^{\mathrm{ad}} = G_1 \times G_2 \times \cdots \times G_r$, where, for each $i = 1, 2, \ldots, r$, G_i is a $\mathbb{Q}$-simple group.

Definition 4.2.2. An **admissible parabolic sub-group** $P \subset G^{\mathrm{ad}}$ is one of the form $P_1 \times P_2 \times \cdots \times P_r$, where, for each i, $P_i \subset G_i$ is a parabolic sub-group. Furthermore, we require:

- For each i, either $P_i = G_i$ or P_i is a maximal proper parabolic sub-group.
- There is at most one i such that $P_i \neq G_i$.

In particular, G^{ad} is an admissible parabolic sub-group of itself.

In general, an **admissible parabolic sub-group** of G is the pre-image of an admissible parabolic sub-group of G^{ad}.

Remark 4.2.3. An admissible parabolic corresponds to a **rational boundary component** F in the the terminology of [AMRT10, §III].

Let $P \subset G$ be an admissible parabolic sub-group and let $U_P \subset P$ be its unipotent radical. Let

$$\cdots \supset (\operatorname{Lie} G)_1 \supset (\operatorname{Lie} G)_0 = \operatorname{Lie} P \supset (\operatorname{Lie} G)_{-1} \supset (\operatorname{Lie} G)_{-2} \supset \cdots$$

be the natural increasing filtration stabilized by P. Then $\operatorname{Lie} U_P = (\operatorname{Lie} G)_{-1}$ and $\operatorname{Lie} U_P^{-2} = (\operatorname{Lie} G)_{-2}$, where U_P^{-2} is the center of U_P. Choose a co-character $w : \mathbb{G}_m \to P$ splitting this natural filtration, and satisfying $ww_0^{-1}(\mathbb{G}_m) \subset G^{\mathrm{der}}$. Here, w_0 is the weight co-character of the Shimura datum (G, X) (cf. 4.1.2), and G^{der} is the derived sub-group of G. Note that w endows every representation V of P with an increasing filtration $W_\bullet V$.

Proposition 4.2.4.

(1) *Given any $x \in X$, and any representation V of G, the filtration $F_x^\bullet V_{\mathbb{C}}$ induced by the map $\mu_x : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ (cf. 4.1.4) determines a rational mixed Hodge structure on V, for which $W_\bullet V$ is the weight filtration. In particular, $F_x^\bullet \operatorname{Lie} P$ endows $\operatorname{Lie} P$ with a polarized mixed Hodge structure of weights $(-1, 1), (0, 0), (1, -1), (0, -1), (-1, 0), (-1, -1)$.*

(2) *For every $x \in X$, there is a canonically associated homomorphism*

$$\varpi_x : \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to P_{\mathbb{C}}$$

splitting the mixed Hodge structure in (1), and whose restriction to the diagonal embedding of $\mathbb{G}_{m,\mathbb{C}}$ in $\mathbb{S}_{\mathbb{C}}$ is conjugate under $P(\mathbb{C})$ to the co-character w.

(3) *Let $Q_P \subset P$ be the smallest normal sub-group such that the maps ϖ_x, as x ranges over X, factor through $Q_{P,\mathbb{R}}$. Let $G_{P,h}$ be the image of Q_P in L_P. If x and x' are in the same connected component of X, then ϖ_x and $\omega_{x'}$ are conjugate under $Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})$. In particular, the assignment $x \mapsto \varpi_x$ maps every connected component X^+ of X into a $Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})$-orbit of co-characters $\mathbb{S}_{\mathbb{C}} \to Q_{P,\mathbb{C}}$. This conjugacy class depends only on the $Q_P(\mathbb{R})$-orbit of X^+ within $\pi_0(X)$, and has a natural holomorphic structure, for which the assignment $x \mapsto \varpi_x$ is holomorphic and $P(\mathbb{R})$-equivariant.*

(4) *Let $Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})$ act on $\pi_0(X)$ via the maps*

$$\pi_0(Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})) = \pi_0(Q_P(\mathbb{R})) \to \pi_0(G(\mathbb{R})).$$

Let

$$F^{(2)}_{P,X^+} \subset \pi_0(X) \times \operatorname{Hom}\bigl(\mathbb{S}_{\mathbb{C}}, Q_{P,\mathbb{C}}\bigr)$$

be the $Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})$-orbit of $\{X^+\} \times \varpi_x$, for any $x \in X^+$ (this does not depend on the choice of x by (3)). Consider the map

$$\begin{aligned} \varphi : X &\to \pi_0(X) \times \operatorname{Hom}\bigl(\mathbb{S}_{\mathbb{C}}, Q_{\mathbb{C}}\bigr) \\ x &\mapsto ([x], \varpi_x), \end{aligned}$$

where $[x]$ denotes the connected component containing X. Then the map $\varphi : \varphi^{-1}\bigl(F^{(2)}_{P,X^+}\bigr) \to F^{(2)}_{P,X^+}$ is an open immersion such that $\pi_0(\varphi)$ is an isomorphism.

(5) *Set $F^{(1)}_{P,X^+} = U_P^{-2}(\mathbb{C})\backslash F^{(2)}_{P,X^+}$ and $F_{P,X^+} = U_P(\mathbb{C})\backslash F^{(2)}_{P,X^+}$. Then $(G_{P,h}, F_{P,X^+})$ is a Shimura datum with reflex field $E(G,X)$.*

Proof. (1) follows from [Bry83, 4.1.5]; cf. also [Pin90, §4] and [AMRT10, §III.4]. In (2) ϖ_x is the map denoted $\omega_x \circ h_\infty$ in [Pin90, 4.6].

For (3) and (4), cf. [Pin90, 4.11]. That $(G_{P,h}, F_{P,X^+})$ is a Shimura datum follows from the description of the mixed Hodge structure on $\operatorname{Lie} P$ in (1), and the assertion about the reflex field can be found in [Pin90, 12.1]. ☐

Remark 4.2.5. The pairs $(Q_{P,X^+}, F^{(2)}_{P,X^+})$ and $(Q_{P,X^+}/U^{-2}_{P,X^+}, F^{(1)}_{P,X^+})$ are **mixed Shimura data** in the terminology of [Pin90, Ch. 2]. The first of these is a **rational boundary component** of (G,X) in the terminology of [Pin90, Ch. 4].

4.2.6. The exponential map gives us an isomorphism of group schemes $\underline{\mathrm{Lie}}(U_P^{-2}) \xrightarrow{\simeq} U_P^{-2}$. From now on, for any $\mathbb{Q}$-algebra R, we will use this isomorphism to identify $U_P^{-2}(R)$ with the free R-module $\operatorname{Lie} U_P^{-2} \otimes_{\mathbb{Q}} R$; in particular, $U_P^{-2}(\mathbb{Q})$ will be a rational pure Hodge structure of weight $(-1,-1)$. We will denote by $U_P^{-2}(\mathbb{Q})(-1)$ the twist of $U_P^{-2}(\mathbb{Q})$ that has weight $(0,0)$.

Recall that, for any $x \in X$, we obtain a Cartan involution $\sigma_x = h_x(i)$ of $G_{\mathbb{R}}$. Let B be the Killing form on $\operatorname{Lie} G$. It follows from [Bry83, 4.1.2] that the pairing $\langle \_,\_\rangle_{\sigma_x} : (v,w) \mapsto B\bigl(v,\sigma_x(w)\bigr)$ induces a positive definite symmetric pairing on $U_P^{-2}(\mathbb{R})$, and hence on $U_P^{-2}(\mathbb{R})(-1)$. Given a connected component $X^+ \subset X$, we have a canonical continuous map $F^{(2)}_{P,X^+} \to U_P^{-2}(\mathbb{R})(-1)$ defined as follows: To every $([x],\varpi) \in F^{(2)}_{P,X^+}$ it attaches the unique element $w \in U_P^{-2}(\mathbb{R})(-1)$ such that $w\varpi w^{-1}$ is defined over $\mathbb{R}$.

Lemma 4.2.7. *There is a canonical open homogeneous self-adjoint (with respect to the pairing $\langle \_,\_\rangle_{\sigma_x}$, for any choice of $x \in X$) convex cone $\mathbf{H}_{P,X^+} \subset U_P^{-2}(\mathbb{R})(-1)$ such that $X^+ \subset F^{(2)}_{P,X^+}$ is the pre-image of $\mathbf{H}_{P,X^+}$ under the map described above.*

Proof. This follows from [Pin90, 4.15]. ☐

Lemma 4.2.8. *Let $P_1, P_2 \subset G$ be two admissible parabolics, and fix $\gamma \in G(\mathbb{Q})$. Then the following statements are equivalent:*

(1) $\gamma U_{P_1}^{-2}\gamma^{-1} \supset U_{P_2}^{-2}$.
(2) $\gamma Q_{P_1}\gamma^{-1} \subset Q_{P_2}$.

Proof. See [AMRT10, III.4.8]. ☐

When P_1, P_2, γ satisfy the equivalent conditions of the lemma above, we denote this situation by $P_1 \xrightarrow{\gamma} P_2$. If, in addition, X_1^+ and X_2^+ are two connected components of X such that $\gamma \cdot X_1^+ = X_2^+$, we will write $(P_1, X_1^+) \xrightarrow{\gamma} (P_2, X_2^+)$. For any admissible parabolic $P \subset G$ and any connected component X^+ of X, let $\mathbf{H}^*_{P,X^+}$ be the union of the cones $\gamma^{-1}\mathbf{H}_{P_2,X_2^+}\gamma \subset U_P^{-2}(\mathbb{R})(-1)$, for all γ, P_2, X_2^+ such that $(P, X^+) \xrightarrow{\gamma} (P_2, X_2^+)$.

4.2.9. Given a Shimura datum (G,X) and a compact open $K \subset G(\mathbb{A}_f)$, we denote by $\mathbf{CLR}_K(G,X)$ the category whose objects are **cusp label representatives** (CLRs for short) **for** (G,X) **of level** K: these are triples (P, X^+, g), where $P \subset G$ is an admissible parabolic, X^+ is a connected component of X, and $g \in G(\mathbb{A}_f)$. A map $(P_1, X_1^+, g_1) \xrightarrow{\gamma} (P_2, X_2^+, g_2)$ is an element $\gamma \in G(\mathbb{Q})$ such that:

- $(P_1, X_1^+) \xrightarrow{\gamma} (P_2, X_2^+)$; cf. the discussion after (4.2.8).

- $\gamma g_1 \in Q_{P_2}(\mathbb{A}_f)g_2K$.

Given an object Φ in $\mathbf{CLR}_K(G,X)$, we will denote by $(P_\Phi, X_\Phi^+, g_\Phi)$ the corresponding tuple. We also have a long list of associated gadgets:

(1) The algebraic groups $Q_\Phi = Q_{P_\Phi}, U_\Phi^{-2} = U_{P_\Phi}^{-2}, G_{\Phi,h} = G_{P_\Phi,h}$.
(2) The spaces $F_\Phi^{(2)} = F^{(2)}_{P_\Phi,X_\Phi^+}, F_\Phi^{(1)} = F^{(1)}_{P_\Phi,X_\Phi^+}, F_\Phi = F_{P_\Phi,X_\Phi^+}$.
(3) The Shimura datum $\big(G_{\Phi,h}, F_\Phi\big)$, and the compact open $K_\Phi \subset G_{\Phi,h}(\mathbb{A}_f)$, obtained as the image of $K_\Phi^{(2)} \coloneqq Q_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1}$.
(4) The Shimura variety $\mathrm{Sh}_{K_\Phi} \coloneqq \mathrm{Sh}_{K_\Phi}\big(G_{\Phi,h}, F_\Phi\big)$ defined over the reflex field $E(G,X)$.
(5) Over Sh_{K_Φ}, there is a natural abelian scheme C_Φ, whose complex points can be identified with the space $Q_\Phi(\mathbb{Q})\backslash F_\Phi^{(1)} \times Q_\Phi(\mathbb{A}_f)/K_\Phi^{(2)}$.
(6) The free abelian groups $\mathbf{B}_\Phi = U_\Phi^{-2}(\mathbb{Q}) \cap g_\Phi K g_\Phi^{-1}$ and $\mathbf{S}_\Phi = \mathbf{B}_\Phi^\vee$, and the $\mathbb{Z}$-torus $\mathbf{E}_\Phi$ with character group $\mathbf{B}_\Phi$.
(7) Over C_Φ, there is a natural $\mathbf{E}_\Phi$-torsor ξ_Φ, whose complex points can be identified with the space $Q_\Phi(\mathbb{Q})\backslash F_\Phi^{(2)} \times Q_\Phi(\mathbb{A}_f)/K_\Phi^{(2)}$.
(8) The homogeneous self-adjoint cone $\mathbf{H}_\Phi = \mathbf{H}_{P_\Phi,X_\Phi^+}$ contained in the cone $\mathbf{H}_\Phi^* = \mathbf{H}^*_{P_\Phi,X_\Phi^+} \subset U_\Phi^{-2}(\mathbb{R})(-1)$.
(9) The discrete sub-group $\Gamma_\Phi \subset \mathrm{Aut}\big(\mathbf{H}_\Phi\big)$: it is the biggest quotient of the group $\mathrm{Aut}(\Phi) = P_\Phi(\mathbb{Q})^+ \bigcap Q_\Phi(\mathbb{A}_f) g_\Phi K g_\Phi^{-1}$ that acts faithfully on $\mathbf{B}_\Phi$. Here, $P_\Phi(\mathbb{Q})^+$ is the stabilizer in $P_\Phi(\mathbb{Q})$ of X_Φ^+.

These are functorial in Φ in the evident ways. In particular, a map $\Phi \xrightarrow{\gamma} \Phi'$ induces an embedding $\gamma^* : \mathbf{H}_{\Phi'}^* \hookrightarrow \mathbf{H}_\Phi^*$ via conjugation by γ^{-1}.

Remark 4.2.10. The varieties C_Φ and ξ_Φ are the (canonical models of) mixed Shimura varieties in the terminology of [Pin90, Ch. 3]. They correspond to the mixed Shimura data seen in (4.2.5).

There is an action of $G(\mathbb{Q})$ on $\mathbf{CLR}_K(G,X)$ given, for $\eta \in G(\mathbb{Q})$, by

$$\eta : \big((P_1, X_1^+, g_1) \xrightarrow{\gamma} (P_2, X_2^+, g_2)\big) \mapsto \big((P_1^\eta, \eta \cdot X_1^+, \eta g_1) \xrightarrow{\eta\gamma\eta^{-1}} (P_2^\eta, \eta \cdot X_2^+, \eta g_2)\big).$$

We also have an action of K on $\mathbf{CLR}_K(G,X)$ given, for $k \in K$, by

$$k : \big((P_1, X_1^+, g_1) \xrightarrow{\gamma} (P_2, X_2^+, g_2)\big) \mapsto \big((P_1, X_1^+, g_1 k) \xrightarrow{\gamma} (P_2, X_2^+, g_2 k)\big).$$

Lemma 4.2.11. *Let $\iota : (G,X) \hookrightarrow (G',X')$ be an embedding of Shimura data. Fix $g \in G'(\mathbb{A}_f)$, and let $K' \subset G'(\mathbb{A}_f)$, $K \subset G(\mathbb{A}_f)$ be compact open sub-groups such that $\iota(K) \subset gK'g^{-1}$. Then:*

(1) There is a canonical functor

$$(\iota, g)_* : \mathbf{CLR}_K(G,X) \to \mathbf{CLR}_{K'}(G',X').$$

(2) Let $\Phi \in \mathbf{CLR}_K(G,X)$ and let $\Phi' = (\iota,g)_*\Phi$. Then the embedding $G \hookrightarrow G'$ induces natural inclusions $U_\Phi \subset U_{\Phi'}$, $U_\Phi^{-2} \subset U_{\Phi'}^{-2}$, $\mathbf{B}_\Phi \subset \mathbf{B}_{\Phi'}$, $\mathbf{H}_\Phi \subset \mathbf{H}_{\Phi'}$, $Q_\Phi \subset Q_{\Phi'}$, $K_\Phi^{(2)} \subset K_{\Phi'}^{(2)}$ and $K_\Phi \subset K_{\Phi'}$.

(3) If $\Phi_1 \xrightarrow{\gamma} \Phi_2$ is a map in $\mathbf{CLR}_K(G,X)$ such that $(\iota,g)_*\gamma$ is an isomorphism, then so is γ.

(4) For $\Phi \in \mathbf{CLR}_K(G,X)$ with $\Phi' = (\iota,g)_*\Phi \in \mathbf{CLR}_{K'}(G',X')$, there is a natural map of mixed Shimura data $(Q_\Phi, F_\Phi^{(2)}) \to (Q_{\Phi'}, F_{\Phi'}^{(2)})$, inducing maps $\xi_\Phi \to \xi_{\Phi'}$, $C_\Phi \to C_{\Phi'}$ and $\mathrm{Sh}_{K_\Phi} \to \mathrm{Sh}_{K'_{\Phi'}}$ of mixed Shimura varieties over $E(G,X)$.

Proof. This is straightforward. The main point is that, for any admissible parabolic sub-group $P \subset G$, there is a unique admissible parabolic $P' \subset G'$ such that $P' \cap G = P$; cf. [Pin90, 4.16].

Given this, the functor $(\iota, g)_*$ sends (P, X^+, h) to $(P', X^{',+}, \iota(h)g)$, where $X^{',+}$ is the connected component of X' containing X^+. The properties in the second assertion are now easily checked.

For (3, the only thing to observe is that, if $\Phi = (P_\Phi, X^+_\Phi, h)$ and $(\iota, g)_*\Phi = (P_{\Phi'}, X^+_{\Phi'}, hg)$, then $P_\Phi = P_{\Phi'} \cap G$.

Finally, (4) is a consequence of [Pin90, 11.10,11.18]. □

Definition 4.2.12. A **cusp label for** (G, X, K) is an isomorphism class of objects in $\mathbf{CLR}_K(G, X)$. We will denote the set of cusp labels for (G, X, K) by $\mathbf{Cusp}_K(G, X)$.

Remark 4.2.13. If, for every CLR Φ, $Q_\Phi(\mathbb{Q})$ acts transitively on the set of connected components of X, then one can easily check that the following definition is equivalent to the one above: A **cusp label for** (G, X, K) is an equivalence class of pairs (P, g), where P is an admissible parabolic sub-group of G, and $g \in G(\mathbb{A}_f)$, where the equivalence relation is as follows: $(P, g) \sim (P', g')$ if there exists $\gamma \in G(\mathbb{Q})$ such that $\gamma P \gamma^{-1} = P'$ and $\gamma g \in Q_{P'}(\mathbb{A}_f)g'K$.

Definition 4.2.14. We will now give a long sequence of definitions that have to do with cone decompositions. See [KKMSD73, §I.2] or [Pin90, §5.1] for further details and any unexplained terminology.

(1) Given $\Phi \in \mathbf{CLR}_K(G, X)$, a **rational polyhedral cone** $\sigma \subset \mathbf{H}^*_\Phi$ is a convex polyhedral cone generated by finitely many elements in $U_\Phi^{-2}(\mathbb{Q})$. We say that σ is **non-degenerate** if it does not contain any lines. We say that σ is **smooth** if it is generated by part of a basis for $\mathbf{B}_\Phi$. Our convention is that all polyhedral cones are closed.

(2) A **rational partial polyhedral cone decomposition** or **rppcd** Σ_Φ for $\mathbf{H}^*_\Phi$ is a collection of rational, non-degenerate, polyhedral cones of $\mathbf{H}^*_\Phi$ such that :
   (*a*) Any face of a cone in Σ_Φ is again a cone in Σ_Φ.
   (*b*) The intersection of any two cones in Σ_Φ is a face of both of them.

   Given an rppcd Σ_Φ for $\mathbf{H}^*_\Phi$, let $\Sigma^\circ_\Phi \subset \Sigma_\Phi$ be the collection of cones σ, whose interior σ° lies in $\mathbf{H}_\Phi$.

(3) An rppcd Σ_Φ for $\mathbf{H}^*_\Phi$ is **smooth** if every cone in Σ_Φ is smooth. It is **complete** if the union of cones in Σ_Φ is all of $\mathbf{H}^*_\Phi$.

(4) An rppcd Σ_Φ for $\mathbf{H}^*_\Phi$ is a **refinement** of another decomposition Σ'_Φ if every cone $\sigma \in \Sigma'_\Phi$ is the union of cones in Σ_Φ that are contained in σ.

(5) A **compatible rppcd** Σ for (G, X, K) is a functorial assignment of an rppcd Σ_Φ to every $\Phi \in \mathbf{CLR}_K(G, X)$. By this, we mean that, for every map $\Phi \xrightarrow{\gamma} \Phi'$,
$$\Sigma_{\Phi'} = (\gamma^*)^{-1}\Sigma_\Phi \coloneqq \{(\gamma^*)^{-1}(\sigma) :\ \sigma \in \Sigma_\Phi\},$$
   where $\gamma^* : \mathbf{H}^*_{\Phi'} \hookrightarrow \mathbf{H}^*_\Phi$ is the induced embedding of cones. We say that Σ is **smooth** (resp. **complete**) if every Σ_Φ is smooth (resp. complete).

(6) A compatible rppcd Σ for (G, X, K) is a **refinement** of another compatible rppcd Σ' if, for every $\Phi \in \mathbf{CLR}_K(G, X)$, Σ_Φ is a refinement of Σ'_Φ.

(7) The disjoint union $\bigsqcup_{\Phi \in \mathbf{CLR}_K(G,X)} \Sigma_\Phi$ has a natural left action by $G(\mathbb{Q})$ and a natural right action by K over the corresponding actions on $\mathbf{CLR}_K(G, X)$. We say that Σ is **admissible** if the double coset space
$$G(\mathbb{Q})\backslash \bigsqcup_{\Phi \in \mathbf{CLR}_K(G,X)} \Sigma_\Phi/K$$
   is *finite*. This is equivalent to requiring that, for each Φ, the number of $\mathrm{Aut}(\Phi)$-orbits in Σ_Φ is finite.

(8) Given an admissible rppcd Σ for (G, X, K), let $\mathbf{Cusp}^\Sigma_K(G, X)$ be the set of equivalence classes of pairs (Φ, σ), where $\Phi \in \mathbf{CLR}_K(G, X)$ and $\sigma \in \Sigma^\circ_\Phi$. Here, we say that two

such pairs (Φ, σ) and (Φ', σ') are **equivalent** if there is an isomorphism $\Phi \xrightarrow[\simeq]{\gamma} \Phi'$ such that $(\gamma^*)^{-1}\sigma = \sigma'$.

(9) Given an admissible Σ and $[(\Phi, \sigma)] \in \mathbf{Cusp}_K^\Sigma(G, X)$, a **face** of $[(\Phi, \sigma)]$ is an equivalence class in $\mathbf{Cusp}_K^\Sigma(G, X)$ of the form $[(\Phi', \sigma')]$, where, for some $\gamma \in G(\mathbb{Q})$, $\Phi' \xrightarrow{\gamma} \Phi$ and σ' is a face of $(\gamma^*)^{-1}(\sigma)$.

(10) Unless otherwise indicated, we will also impose the following condition on admissible rppcds Σ (cf. [Lan08, 6.5.2.25];[Pin90, 7.12]): Given $\Phi \in \mathbf{CLR}_K(G, X)$ and $\sigma \in \Sigma_\Phi$, let $\Phi \xrightarrow{\gamma} \Phi'$ be a map such that σ is in the image of $\gamma^* : \mathbf{H}_{\Phi'} \subset \mathbf{H}_{\Phi'}$. Then we require any automorphism $\eta \in \Gamma_\Phi$ with $\eta \cdot \sigma \cap \sigma \neq \emptyset$ to act trivially on $\gamma^*(\mathbf{H}_\Phi)$.

Remark 4.2.15. Our definition of an admissible rppcd is stricter than the ones found in [Har89] and [Pin90] (which are themselves slightly different from each other). In particular, our 'admissible' is Pink's 'finite admissible' in [Pin90]. Our definition, however, agrees with the one found in [Lan08] in the PEL case.

Remark 4.2.16. We know from [AMRT10, Ch. II] and [Pin90, Ch. 9] that complete admissible compatible rppcds exist. Moreover, every admissible rppcd admits a smooth refinement, and any two admissible rppcds admit a common refinement. For any prime p, we can ensure that condition (10) is valid by shrinking the prime-to-p part K^p of K; cf. [Pin90, 7.13].

Definition 4.2.17. We will say that a CLR Φ for (G, X, K) is **improper** if $P_\Phi = G$. A cusp label $[\Phi]$ is **improper** if it is the class of an improper CLR. For an improper CLR Φ, the unipotent radical U_Φ, and hence the objects U_Φ^{-2} and $\mathbf{H}_\Phi$, are trivial. Moreover, $H \coloneqq Q_\Phi \subset G$ is the smallest normal $\mathbb{Q}$-rational sub-group generated by X, and is reductive. The improper cusp labels are in bijection with the double coset space $G(\mathbb{Q})H(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K$, and given such a cusp label $[\Phi]$, and any admissible rppcd Σ, we will also denote by $[\Phi]$ the unique class in $\mathbf{Cusp}_K^\Sigma(G, X)$ that it gives rise to.

4.2.18. Let Σ be an admissible rppcd for (G, X, K), and let $\mathrm{Sh}_K^\Sigma(G, X)$ be the associated partial toroidal compactification of $\mathrm{Sh}_K(G, X)$. Suppose Φ and Φ' are representatives of the same class in $\mathbf{Cusp}_K(G, X)$, and let $\gamma \in G(\mathbb{Q})$ be an element such that $\Phi \xrightarrow{\gamma} \Phi'$. Conjugation by γ gives a morphism $\mathrm{int}(\gamma) : (Q_\Phi, F_\Phi^{(2)}) \to (Q_{\Phi'}, F_{\Phi'}^{(2)})$ of mixed Shimura data. Suppose that $q \in Q_{\Phi'}(\mathbb{A}_f)$ is such that $\gamma g_\Phi \in q g_{\Phi'} K$; then $\mathrm{int}(\gamma)(K_\Phi) = q K_{\Phi'} q^{-1}$. We therefore get a map $[\gamma, q] : \xi_\Phi \to \xi_{\Phi'}$ of mixed Shimura varieties. On the level of complex points, for $(\omega, g) \in F_\Phi^{(2)} \times Q_\Phi(\mathbb{A}_f)$, we have $[\gamma, q]([(\omega, g)]) = [(\gamma \cdot \omega, \mathrm{int}(\gamma)(g)q)]$; that this map descends to a map of varieties over $E(G, X)$ follows from [Pin90, 11.10]. It is easy to check from its explicit description over $\mathbb{C}$ that $[\gamma, q]$ depends only on γ and not on the choice of q; we will therefore denote it simply by $[\gamma]$.

Theorem 4.2.19 (Ash-Mumford-Rapoport,Pink)**.** *Assume now that K is* neat. *Given any admissible rppcd Σ for (G, X, K), there exists an algebraic space $\mathrm{Sh}_K^\Sigma(G, X)$ over $E(G, X)$ containing $\mathrm{Sh}_K(G, X)$ as an open dense sub-variety and satisfying the following properties:*

(1) The complement D_K^Σ of $\mathrm{Sh}_K(G, X)$ in $\mathrm{Sh}_K^\Sigma(G, X)$ is an effective Cartier divisor, along which $\mathrm{Sh}_K^\Sigma(G, X)$ has at most toroidal singularities.

(2) If Σ is complete, then $\mathrm{Sh}_K^\Sigma(G, X)$ is proper; if Σ is smooth, then $\mathrm{Sh}_K^\Sigma(G, X)$ is smooth.

(3) There is a stratification by smooth locally closed sub-varieties

$$\mathrm{Sh}_K^\Sigma(G, X) = \bigsqcup_{[(\Phi, \sigma)]} Z_{[(\Phi, \sigma)]},$$

where $[(\Phi, \sigma)]$ ranges over $\mathbf{Cusp}_K^\Sigma(G, X)$. In this stratification, $Z_{[(\Phi, \sigma)]}$ is in the closure of $Z_{[(\Phi', \sigma')]}$ if and only if $[(\Phi', \sigma')]$ is a face of $[(\Phi, \sigma)]$. In particular, the strata of the form $Z_{[\Phi]}$, for $[\Phi]$ improper are open and closed in $\mathrm{Sh}_K^\Sigma(G, X)$.

(4) *For every $[(\Phi,\sigma)] \in \mathbf{Cusp}_K^{\Sigma}(G,X)$ with representative (Φ,σ), $Z_{[(\Phi,\sigma)]}$ is canonically isomorphic to the closed stratum $Z_\Phi(\sigma)$ in the twisted torus embedding $\xi_\Phi(\sigma) = \xi_\Phi \times^{\mathbf{E}_\Phi} \mathbf{E}_\Phi(\sigma)$. In fact, the completion of $\mathrm{Sh}_K^{\Sigma}(G,X)$ along $Z_{[(\Phi,\sigma)]}$ is canonically isomorphic to the completion of $\xi_\Phi(\sigma)$ along $Z_\Phi(\sigma)$. In particular, $\mathrm{Sh}_K(G,X)$ itself is the union of the strata indexed by the improper cusp labels.*

(5) *Suppose that (Φ,σ) and (Φ',σ') are two representatives of a class in $\mathbf{Cusp}_K^{\Sigma}(G,X)$, and let $\gamma \in G(\mathbb{Q})$ be such that $\Phi \xrightarrow{\gamma} \Phi'$ and $\gamma^*\sigma' = \sigma$. Then the following diagram commutes:*

$$\begin{tikzcd} & Z_{[(\Phi,\sigma)]} \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ Z_\Phi(\sigma) \arrow[rr, "{[\gamma]}", "\simeq"'] & & Z_{\Phi'}(\sigma'). \end{tikzcd}$$

Here the isomorphism $[\gamma]$ in the bottom row is the one induced from the isomorphism $[\gamma] : \xi_\Phi \to \xi_{\Phi'}$ discussed in (4.2.18), and the diagonal maps are the isomorphisms from (4).

Proof. This follows from [Pin90, 12.4]. □

Lemma 4.2.20. *Suppose that we are given an embedding $\iota : (G,X) \hookrightarrow (G',X')$ of Shimura data. Let K, K', g be as in (4.2.11), so that there is a functor*

$$(\iota,g)_* : \mathbf{CLR}_K(G,X) \to \mathbf{CLR}_{K'}(G',X').$$

(1) *Every compatible rppcd Σ' for (G',X',K') naturally gives rise to a compatible rppcd $\Sigma = (\iota,g)^*\Sigma'$ for (G,X,K).*
(2) *If Σ' is admissible (resp. complete), then Σ is also admissible (resp. complete).*
(3) *If Σ is any admissible rppcd for (G,X,K), then we can find an admissible rppcd Σ' for (G',X',K') such that $(\iota,g)^*\Sigma'$ is a refinement of Σ.*
(4) *In (3), we can choose Σ' such that both Σ' and $(\iota,g)^*\Sigma'$ are smooth.*

Proof. For $\Phi \in \mathbf{CLR}_K(G,X)$ with $\Phi' = (\iota,g)_*\Phi \in \mathbf{CLR}_K(G,X)$, there is a natural embedding of groups $U_\Phi^{-2} \subset U_{\Phi'}^{-2}$ inducing an embedding cones $\mathbf{H}_\Phi^* \subset \mathbf{H}_{\Phi'}^*$. So the rational cone decomposition $\Sigma_{\Phi'}$ will determine a rational cone decomposition Σ_Φ for $\mathbf{H}_\Phi^*$. This gives us the induced compatible rppcd $\Sigma = (\iota,g)^*\Sigma'$ in (1).

As for (2), the inheritance of completeness by Σ is clear. The inheritance of admissibility follows from [Har89, 3.3].

We will only indicate the idea of the proofs of (3) and (4); cf. also [Hör10, 2.4.12]. For (3), we proceed as follows: For each $\Phi' \in \mathbf{CLR}_K(G,X)$, we will define a new cone decomposition $\Sigma'_{\Phi'}$ for $\mathbf{H}_{\Phi'}^*$ as follows: it will consist of the intersection of the cones in Σ_Φ with the cones in Σ_Φ along the inclusions $\mathbf{H}_\Phi^* \hookrightarrow \mathbf{H}_{\Phi'}^*$ associated with any $\Phi \in \mathbf{CLR}_K(G,X)$ such that $\Phi' = (\iota,g)_*\Phi$. One can check now that the $\Sigma'_{\Phi'}$ defined in this way patch together to give an admissible rppcd for (G',X',K'); clearly, $(\iota,g)^*\Sigma'$ refines Σ. Finally, for (4), take any smooth refinement Σ'' of the just constructed Σ'. Then, by construction, $(\iota,g)^*\Sigma''$ will also be smooth. □

Proposition 4.2.21. *Let $\iota : (G,X) \hookrightarrow (G',X')$ be an embedding of Shimura data, and let K, K', g be as in (4.2.20). Suppose that both K and K' are neat. Let Σ' be an admissible rppcd for (G',X',K') and let Σ be an admissible rppcd for (G,X,K) refining the one induced from Σ' via (4.2.20)(1).*

(1) *There exists a natural map $[g]_{K,K'}^{\Sigma,\Sigma'} : \mathrm{Sh}_K^{\Sigma}(G,X) \to \mathrm{Sh}_{K'}^{\Sigma'}(G',X')$ extending the map $[g]_{K,K'} : \mathrm{Sh}_K(G,X) \hookrightarrow \mathrm{Sh}_{K'}(G',X')$ induced by the map (ι,g) of Shimura data.*

(2) Under the map in (1), for any $[(\Phi,\sigma)] \in \mathbf{Cusp}^{\Sigma}_K(G,X)$*, the stratum* $Z_{[(\Phi,\sigma)]}$ *maps into the stratum* $Z_{[(\Phi',\sigma')]}$*, where* $[(\Phi',\sigma')] \in \mathbf{Cusp}^{\Sigma'}_{K'}(G',X')$ *is determined in the following way:* $\Phi' = (\iota,g)_*\Phi$ *and* $\sigma' \in \Sigma'_{\Phi'}$ *is the minimal cone that contains* σ*.*

(3) For (Φ,σ) *and* (Φ',σ') *as in (2), the map* $[g]_{[(\Phi,\sigma)]} : Z_{[(\Phi,\sigma)]} \to Z_{[(\Phi',\sigma')]}$ *can be described as follows: The natural map of mixed Shimura varieties* $\xi_\Phi \to \xi_{\Phi'}$ *is a* $\mathbf{E}_\Phi$*-equivariant map over the natural map* $C_\Phi \to C_{\Phi'}$*. Now,* $[g]_{[(\Phi,\sigma)]}$ *is isomorphic to the map between the closed stratum* $Z_\Phi(\sigma) \subset \xi_\Phi(\sigma)$ *into the closed stratum of* $Z_{\Phi'}(\sigma') \subset \xi_{\Phi'}(\sigma')$*. Similarly, the induced map* $\widehat{[g]}_{[(\Phi,\sigma)]}$ *on the completion of* $\mathrm{Sh}^{\Sigma}_K(G,X)$ *along* $Z_{[(\Phi,\sigma)]}$ *is isomorphic to the natural map between the completions of* $\xi_\Phi(\sigma)$ *and* $\xi_{\Phi'}(\sigma')$ *along their closed strata.*

Proof. See [Pin90, 6.25,12.4]. □

4.3. **Chai-Faltings compactifications.**

4.3.1. Suppose $(G,X) = (\mathrm{GSp},\mathrm{S}^{\pm})$ is the Siegel Shimura datum associated with a symplectic space (V,ψ) over $\mathbb{Q}$. In this case, any maximal parabolic $P \subset G$ is the stabilizer of an isotropic sub-space $W \subset V$; or, equivalently, of the filtration

$$0 = W_{-3}V \subset W_{-2}V = W \subset W_{-1}V = W^{\perp} \subset W_0V = V.$$

X is the union of two connected components, and the choice of connected component can be seen as the choice of an isomorphism of vector spaces $\mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}(1)$; in other words, as a choice of orientation for $\mathbb{C}$. In particular, choosing a connected component X^+ of X gives us an isomorphism $U_P^{-2}(\mathbb{Q})(-1) \xrightarrow{\simeq} U_P^{-2}(\mathbb{Q})$.

We now see (cf. [Mor08, 1.2]):

- $U_P^{-2} \subset P$ is the sub-group of elements acting trivially on $W_{-1}V$ and can be identified with the group of homomorphisms $V/W_{-1}V \to W_{-2}V$, which are symmetric with respect to the identification $W_{-2}V = \big(V/W_{-1}V\big)^{\vee}$ induced by ψ. In particular, we can identify $U_P^{-2}(\mathbb{Q})$ with the group of symmetric bilinear pairings on $V/W_{-1}V$.
- L_P is identified with the sub-group of $\mathrm{GL}\big(W_{-2}V\big)\times\mathrm{GSp}\big(\mathrm{gr}^W_{-1} V\big)\times\mathrm{GL}\big(\mathrm{gr}^W_0 V\big)$ consisting of elements (g_1,g_2,g_3), where $g_1 = g_3^t$ under the identification $W_{-2}V = \big(V/W_{-1}V\big)^{\vee}$.
- $Q_P \subset P$ is the normal sub-group of elements acting trivially on $\mathrm{gr}^W_0 V$: it acts transitively on the connected components of $\mathrm{S}^{\pm}$.
- $G_{P,h} = \mathrm{GSp}\big(\mathrm{gr}^W_0 V\big)$, and the Shimura datum $\big(G_{P,h},X_P\big)$ is simply the Siegel Shimura datum associated with the symplectic space $\big(\mathrm{gr}^W_{-1} V, \mathrm{gr}^W_{-1}\psi\big)$ (with the agreed upon meaning when $\mathrm{gr}^W_{-1} V = 0$; cf. 4.1.6).
- Given a connected component X^+, the cone $\mathbf{H}_{P,X^+} \subset U_P^{-2}(\mathbb{R})(-1)$ is the pre-image of the cone of positive definite symmetric pairings on $V/W_{-1}V$ under the isomorphism $U_P^{-2}(\mathbb{Q})(-1) \xrightarrow{\simeq} U_P^{-2}(\mathbb{Q})$ afforded by the choice of X^+. $\mathbf{H}^*_{P,X^+}$ is the pre-image of the cone of positive semi-definite symmetric pairings on $V/W_{-1}V$.

4.3.2. Let $V_{\mathbb{Z}} \subset V$ be a polarized $\mathbb{Z}$-lattice with discriminant d^2. We would like to reconcile the present notion of cusp labels with the one introduced in (3.1.7). Fix $n \in \mathbb{Z}_{>0}$, and set

$$K(n) = \mathrm{GSp}(\mathbb{A}_f)\bigcap\ker\Big(\mathrm{GL}\big(V_{\mathbb{Z}}\otimes\widehat{\mathbb{Z}}\big) \to \mathrm{GL}\big(V_{\mathbb{Z}}\otimes(\mathbb{Z}/n\mathbb{Z})\big)\Big) \subset \mathrm{GSp}(\mathbb{A}_f).$$

This is the **level** n **sub-group associated with** $V_{\mathbb{Z}}$. Let $\mathbf{M}_{V_{\mathbb{Z}},n,\psi}$ be the moduli space from (3.1.1); then its fiber over $\mathbb{Q}$ is canonically identified with $\mathrm{Sh}_{K(n)}\big(\mathrm{GSp}(V),\mathrm{S}^{\pm}\big)$. From now on, we will write $\mathscr{S}_{K(n)}$ for the fiber of $\mathbf{M}_{V_{\mathbb{Z}},n,\psi}$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$.

Lemma 4.3.3.

(1) *There is a bijection between the set of cusp labels for* $(\mathrm{GSp},\mathrm{S}^{\pm},K(n))$ *defined in (4.2.12) and the set of cusp labels for* $(V_{\mathbb{Z}},\psi)$ *at level* n *defined in (3.1.7).*
(2) *Let* $[\Phi]$ *be a cusp label for* $(\mathrm{GSp},\mathrm{S}^{\pm},K(n))$*, and let* $[(\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}},\Phi',\delta)]$ *be the corresponding cusp label for* $(V_{\mathbb{Z}},\psi)$ *at level* n*. Then the free abelian groups* $\mathbf{B}_\Phi$ *from (4.2.9) and* $\mathbf{B}_{\Phi'}$ *from (3.1.13) are naturally identified. In particular, the tori* $\mathbf{E}_\Phi$ *and* $\mathbf{E}_{\Phi'}$ *are naturally identified.*
(3) *Let* $[\Phi]$ *and* $[(\mathrm{W}_\bullet V_{\mathbb{Z}/n\mathbb{Z}},\Phi',\delta)]$ *be as above. Then the fiber over* $\mathbb{Q}$ *of the tower*

$$\mathbf{\Xi}_{\Phi',\delta}\to\mathbf{C}_{\Phi',\delta}\to\mathbf{M}_{\mathrm{W}_\bullet}$$

of an $\mathbf{E}_\Phi$*-torsor over an abelian scheme over* $\mathbf{M}_{\mathrm{W}_\bullet}$ *considered in § 3.1 is naturally isomorphic to the analogous tower* $\xi_\Phi\to C_\Phi\to\mathrm{Sh}_{K(n)_\Phi}$ *from (4.2.19)(4).*

Proof. By (4.3.1) and (4.2.13), a cusp label for $(G,X,K(n))$ is an equivalence class of pairs $(W_\bullet V,g)$, where $W_\bullet V$ is a three step filtration of V:

$$0=W_{-3}V\subset W_{-2}V\subset W_{-1}V=\bigl(W_{-2}V\bigr)^{\perp}\subset W_0V=V;$$

and $g\in\mathrm{GSp}(\mathbb{A}_f)$. Under this equivalence relation, $(W_\bullet V,g)$ and $(W'_\bullet V,g')$ are equivalent if there exists $\gamma\in\mathrm{GSp}(\mathbb{Q})$ such that $\gamma(W_\bullet V)=W'_\bullet V$, and if $\gamma g\in Q_{P'}(\mathbb{A}_f)g'K(n)$, where $P'\subset\mathrm{GSp}$ is the parabolic sub-group stabilizing $W'_\bullet V$ and $Q_{P'}\subset P'$ is as in (4.3.1).

Given a pair $(W_\bullet V,g)$, we can define an associated torus argument Φ' as in (3.1.4) for the induced filtration $W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$ as follows (cf. also [Lan10a, §3.1]): Set $V_{\mathbb{Z}}^{(g)}=g\cdot V_{\mathbb{Z}}\subset V_{\mathbb{A}_f}$. We take $Y=\mathrm{gr}^W_0V_{\mathbb{Z}}^{(g)}$ and $X=\mathrm{Hom}\bigl(W_2V_{\mathbb{Z}}^{(g)},\mathbb{Z}\bigr)$; $\lambda^{\text{ét}}:Y\to X$ will be the map induced by the pairing between gr^W_0V and W_2V; the maps $\varphi_n^{\text{ét}}$ and φ_n^{mult} will just be the reduction mod-n of the isomorphisms $\mathrm{gr}^W_\bullet V_{\mathbb{Z}}\xrightarrow{\simeq}\mathrm{gr}^W_\bullet V_{\mathbb{Z}}^{(g)}$ induced by multiplication by g. The cusp label (in the sense of (3.1.7)) associated with the pair $(W_\bullet V,g)$ will now be represented by $(W_\bullet V_{\mathbb{Z}/n\mathbb{Z}},\Phi',\delta)$, where δ is any splitting of $W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$. It is now easy to check that this assignment defines a bijection as in (1), and that it satisfies (2).

Finally, (3) is a consequence of [Lan10a, 3.6.11]. Note that $\mathbf{M}_{\mathrm{W}_\bullet}$ has the meaning explained in (3.1.2) when $\mathrm{gr}^{\mathrm{W}}_{-1}V_{\mathbb{Z}/n\mathbb{Z}}=0$. □

Remark 4.3.4. In the situation of (3) above, we will write the tower $\mathbf{\Xi}_{\Phi',\delta}\to\mathbf{C}_{\Phi',\delta}\to\mathbf{M}_{\mathrm{W}_\bullet}$ as $\mathbf{\Xi}_\Phi\to\mathbf{C}_\Phi\to\mathscr{S}_{K(n)_\Phi}$. Implicit in this notation is the fact that the tower only depends on the cusp label $[\Phi]$ for $(\mathrm{GSp},\mathrm{S}^{\pm},K(n))$.

4.3.5. Suppos that we have $\Phi\in\mathrm{CLR}_{K(n)}(\mathrm{GSp},\mathrm{S}^{\pm})$ and $\sigma\subset\mathbf{H}_\Phi$. The description in (3.1.13) shows that over $\mathbf{\Xi}_\Phi(\sigma)$ there is a canonical polarized log 1-motif $(\mathrm{Q}_{(\Phi,\sigma)},\lambda_{(\Phi,\sigma)})$ attached to the tautological tuple

$$(\mathrm{B},Y,X,c_\Phi,c_\Phi^\vee,\lambda^{\mathrm{ab}},\lambda^{\text{ét}},\tau_{(\Phi,\sigma)}).$$

The fact that σ lies within $\mathbf{H}_\Phi$ implies that the polarization is in fact positive.

Given an affine open $\operatorname{Spec}R\subset\mathbf{\Xi}_\Phi(\sigma)$, let $\operatorname{Spf}\widehat{R}$ be its completion along the closed stratum of $\mathbf{\Xi}_\Phi(\sigma)$. Equip $\operatorname{Spec}\widehat{R}$ with the induced log structure, and let $U\subset\operatorname{Spec}\widehat{R}$ be the complement of the boundary divisor; then by (2.3.4), the positively polarized log 1-motif $(\mathrm{Q}_{(\Phi,\sigma)},\lambda_{(\Phi,\sigma)})$ gives rise to a polarized abelian scheme (A,λ) over U that extends to a semi-abelian scheme over $\operatorname{Spec}\widehat{R}$.

Moreover, there exist: a tautological full level-n structure on B, and liftings $c_{\Phi,n}$, $c^\vee_{\Phi,n}$ and $\tau_{(\Phi,\sigma),n}$ of c_Φ, $c_\Phi^\vee$ and $\tau_{(\Phi,\sigma)}$, respectively. These combine to equip (A,λ) with a full level-n structure and determine a canonical morphism $U\to\mathscr{S}_{K(n)}$.

4.3.6. What follows is the main global theorem in the theory of integral toroidal compactifications.

Theorem 4.3.7 (Chai-Faltings,Lan)**.** *Fix $n \geq 3$ and let $K = K(n)$. Given a smooth admissible rppcd Σ for $(\mathrm{GSp}, \mathrm{S}^{\pm}, K)$, there exists a smooth algebraic space $\mathscr{S}_K^{\Sigma} = \mathscr{S}_K^{\Sigma}(\mathrm{GSp}, \mathrm{S}^{\pm})$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$ containing $\mathscr{S}_K$ as a open dense sub-scheme and satisfying the following properties:*

(1) *The boundary $\mathbf{D}_K^{\Sigma} = \mathscr{S}_K^{\Sigma} \backslash \mathscr{S}_K$ is a relative effective Cartier divisor with normal crossings, and $\mathscr{S}_K^{\Sigma}$, equipped with the associated log structure, is log smooth over $\mathbb{Z}\left[\frac{1}{nd}\right]$.*

(2) *If Σ is complete, then $\mathscr{S}_K^{\Sigma}$ is proper over $\mathbb{Z}\left[\frac{1}{nd}\right]$.*

(3) *The generic fiber $\mathscr{S}_K^{\Sigma} \otimes \mathbb{Q}$ is naturally isomorphic to the partial toroidal compactification $\mathrm{Sh}_K^{\Sigma}(\mathrm{GSp}, \mathrm{S}^{\pm})$ of $\mathrm{Sh}_K(\mathrm{GSp}, \mathrm{S}^{\pm})$ defined in (4.2.19).*

(4) *$\mathscr{S}_K^{\Sigma}$ admits a stratification by smooth sub-schemes:*
$$\mathscr{S}_K^{\Sigma} = \bigsqcup_{[(\Phi,\sigma)]} \mathbf{Z}_{[(\Phi,\sigma)]},$$
where $[(\Phi, \sigma)]$ ranges over $\mathbf{Cusp}_K^{\Sigma}(\mathrm{GSp}, \mathrm{S}^{\pm})$. This is compatible with the stratification of its generic fiber in (4.2.19)(3).

(5) *For every $[(\Phi, \sigma)] \in \mathbf{Cusp}_K^{\Sigma}(\mathrm{GSp}, \mathrm{S}^{\pm})$ with representative (Φ, σ), $\mathbf{Z}_{[(\Phi,\sigma)]}$ is canonically isomorphic to the closed stratum $\mathbf{Z}_{\Phi}(\sigma)$ in the twisted torus embedding $\mathbf{\Xi}_{\Phi}(\sigma)$. In fact, the completion $\mathfrak{X}_{[(\Phi,\sigma)]}$ of $\mathscr{S}_K^{\Sigma}$ along $\mathbf{Z}_{[(\Phi,\sigma)]}$ is canonically isomorphic to the completion of $\mathbf{\Xi}_{\Phi}(\sigma)$ along $\mathbf{Z}_{\Phi}(\sigma)$.*

(6) *Every complete Σ admits a refinement Σ' such that $\mathscr{S}_K^{\Sigma'}$ is projective.*

Proof. Assertions (1), (2), (4) and (5) follow from [Lan08, 6.4.1.1]; cf. also [FC90, §V.2]. Let $\mathfrak{X}_{\Phi}(\sigma)$ be the completion of $\mathbf{\Xi}_{\Phi}(\sigma)$ along its closed stratum. The canonical strata preserving map
$$j : \mathfrak{X}_{\Phi}(\sigma) \to \mathfrak{X}_{[(\Phi,\sigma)]}$$
implicit in the statement of (5) is characterized by the following property: As in (4.3.5), suppose that we are given an affine open $\operatorname{Spec} R \subset \mathbf{\Xi}_{\Phi}(\sigma)$ with completion $\operatorname{Spf} \widehat{R}$ along the closed stratum. Let $U \subset \operatorname{Spec} \widehat{R}$ be the complement of the boundary divisor. Then the restriction of j to $\operatorname{Spf} \widehat{R}$ is the unique map arising from the canonical map $U \to \mathscr{S}_{K(n)}$ described in *loc. cit.*

As for (3), this follows from [Lan10a, 4.1.1], though some care must be taken to descend the cited assertion from $\mathbb{C}$ down to the reflex field. To do this, we take the Zariski closure Δ^{tor} of the diagonal $\Delta \subset \mathrm{Sh}_K(\mathrm{GSp}, \mathrm{S}^{\pm}) \times \mathrm{Sh}_K(\mathrm{GSp}, \mathrm{S}^{\pm})$ in $\mathrm{Sh}_K^{\Sigma}(\mathrm{GSp}, \mathrm{S}^{\pm}) \times \left(\mathscr{S}_K^{\Sigma} \otimes \mathbb{Q}\right)$. We have to check that Δ^{tor} is the graph of an isomorphism; that is, it maps isomorphically onto both factors. This can be done over $\mathbb{C}$, which is precisely what is accomplished in the proof of *loc. cit.*

Note that in [FC90, Lan08, Lan10a], all admissible rppcds Σ are assumed to be complete, but the construction, and its comparison with the analytic construction, go through for any smooth admissible rppcd.

Finally, (6) follows from [FC90, V.5.8] (cf. also [Lan08, 7.3.3.4]). ☐

4.4. **Intersection with the boundary and Morita's conjecture.**

4.4.1. Fix a Shimura datum (G, X) and an embedding $\iota : (G, X) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm})$. Let (V, ψ) be the symplectic space to which GSp is attached. Fix a polarized $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$, and let $d = \sharp\left(V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}\right)$ be its discriminant: we will assume that $(p, d) = 1$. For every integer n such that $(n, pd) = 1$, let $K'(n) \subset \mathrm{GSp}(\mathbb{A}_f)$ be the sub-group of level n associated with $V_{\mathbb{Z}}$, and let $K(n) = K'(n) \cap G(\mathbb{A}_f)$.

For $n \geq 3$ large enough, it follows from (4.1.5) that there is a closed embedding of Shimura varieties $\mathrm{Sh}_{K(n)}(G, X) \hookrightarrow \mathrm{Sh}_{K'(n)}(\mathrm{GSp}, \mathrm{S}^{\pm})_E$ over the reflex field $E = E(G, X)$. Note that $\mathrm{Sh}_{K'(n)}(\mathrm{GSp}, \mathrm{S}^{\pm})$ is just the fine moduli space $\mathbf{M}_{V_{\mathbb{Z}/n\mathbb{Z}},\psi} \otimes_{\mathbb{Z}[1/n]} \mathbb{Q}$ introduced in (3.1.1), and as

such has the canonical integral model $\mathscr{S}_{K'(n)}(\mathrm{GSp},\mathrm{S}^{\pm}) = \mathbf{M}_{V_{\mathbb{Z}/n\mathbb{Z}},\psi}$ over $\mathbb{Z}\left[\frac{1}{dn}\right]$. Let $v|p$ be a prime of E lying above p; let E_v be the completion of E along v, and let $\mathscr{O}_{E,(v)}$ be its ring of integers. Let $\breve{\mathscr{S}}_{K(n)}(G,X)$ be the Zariski closure of $\mathrm{Sh}_{K(n)}(G,X)$ in $\mathscr{S}_{K'(n)}(\mathrm{GSp},\mathrm{S}^{\pm})_{\mathscr{O}_{E,(v)}}$. To keep notation light, we will write K' for $K'(n)$, K for $K(n)$, and we will contract the notation for the $\mathscr{O}_{E,(v)}$-schemes $\mathscr{S}_{K'(n)}(\mathrm{GSp},\mathrm{S}^{\pm})_{\mathscr{O}_{E,(v)}}$ and $\breve{\mathscr{S}}_{K(n)}(G,X)$ to $\mathscr{S}_{K'}$ and $\breve{\mathscr{S}}_{K}$, respectively. Similarly, we will denote $\mathrm{Sh}_{K'(n)}(\mathrm{GSp},\mathrm{S}^{\pm})_E$ and $\mathrm{Sh}_{K(n)}(G,X)$ by $\mathrm{Sh}_{K'}$ and Sh_K, respectively.

Fix $\Phi \in \mathbf{CLR}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$, a rational polyhedral cone $\sigma \subset \mathbf{H}_\Phi$, and suppose that x_0 is a closed point in the closed stratum $\mathbf{Z}_\Phi(\sigma)_{\mathscr{O}_{E,(v)}} \subset \mathbf{\Xi}_\Phi(\sigma)_{\mathscr{O}_{E,(v)}}$. We will assume that x_0 is valued in a finite field k of characteristic p. Let R_{Φ,σ,x_0} be the complete local ring of $\mathbf{\Xi}_\Phi(\sigma)$ at x_0, and let $R_{\Phi,\sigma,x_0,v}$ be its base change over $\mathscr{O}_{E_v}$. By the discussion in (4.3.5), the complement $U^\circ_{\Phi,\sigma,x_0,v}$ of the boundary divisor in $\operatorname{Spec} R_{\Phi,\sigma,x_0,v}$ admits a canonical map to $\mathscr{S}_{K'}$.

Abusing notation, we will write $\breve{\mathscr{S}}_K \cap U^\circ_{\Phi,\sigma,x_0}$ for the pull-back of $\breve{\mathscr{S}}_K$ over $U^\circ_{\Phi,\sigma,x_0}$.

Definition 4.4.2. We will say that $\operatorname{Spec} R_{\Phi,\sigma,x_0,v}$ **intersects** $\breve{\mathscr{S}}^\Sigma_K$ at x_0 if the Zariski closure of the image of $\breve{\mathscr{S}}_K \cap U^\circ_{\Phi,\sigma,x_0}$ in $\operatorname{Spec} R_{\Phi,\sigma,x_0,v}$ contains the closed point.

In this case, we will write T_{Φ,σ,x_0} for the quotient of $R_{\Phi,\sigma,x_0,v}$ attached to this Zariski closure.

4.4.3. Let Φ,σ and x_0 be as above, and suppose that $k(x_0)$ is a finite field of characteristic p. As in (3.2), denote by (Q_0,λ_0) the induced polarized log 1-motif over $k(x_0)$. Let $L/W(k)_{\mathbb{Q}}$ be a finite extension, and let $x : \operatorname{Spec}\mathscr{O}_L \to \mathbf{\Xi}_\Phi(\sigma)$ be a lift of x_0 carrying the generic point into the complement of $\mathbf{Z}_\Phi(\sigma)$. Then we have the induced polarized log 1-motif (Q_x,λ_x) over $\mathscr{O}_L$. Also, by the above discussion, $x|_{\operatorname{Spec} L}$ can be canonically viewed as a point of $\mathscr{S}_K$, and if $\mathcal{A}_x$ is the attached abelian variety over L, it follows from (2.3.4) that there exists a canonical isomorphism of filtered L-vector spaces:

$$H^1_{\mathrm{dR}}(\mathcal{A}_x/L) \xrightarrow{\simeq} H^1_{\mathrm{dR}}(Q_x) \otimes_{\mathscr{O}_L} L.$$

Similarly, if we fix an algebraic closure $\overline{L}/L$, there exists a canonical isomorphism of $\mathrm{Gal}(\overline{L}/L)$-modules:

$$H^1_{\text{ét}}\big(\mathcal{A}_{x,\overline{L}},\mathbb{Z}_p\big) \xrightarrow{\simeq} H^1_{\text{ét}}\big(Q_{x,\overline{L}},\mathbb{Z}_p\big).$$

Fix a uniformizer $\pi \in \mathscr{O}_L$, and let $W = W(k)$; then (2.4.10) shows that we have natural comparison isomorphisms:

$$\mathbb{D}(Q_0)(W_{\Phi,\sigma}) \otimes_W L \xrightarrow{\simeq} H^1_{\mathrm{dR}}(\mathcal{A}_x/L). \tag{4.4.3.1}$$

$$\mathbb{D}(Q_0)(W_{\Phi,\sigma}) \otimes_W B_{\mathrm{st}} \xrightarrow{\simeq} H^1_{\text{ét}}\big(\mathcal{A}_{x,\overline{L}},\mathbb{Z}_p\big) \otimes_{\mathbb{Z}_p} B_{\mathrm{st}} \tag{4.4.3.2}$$

4.4.4. Choose tensors $\{s_\alpha\} \subset V^\otimes$ such that $G \subset \mathrm{GSp}$ is their pointwise stabilizer. Let $\boldsymbol{V}_{\mathrm{dR},E}$ be the relative first de Rham cohomology of the universal abelian scheme $\mathcal{A}$ over Sh_K. By (4.1.10), we now obtain parallel Hodge tensors over Sh_K:

$$\{s_{\alpha,\mathrm{dR}}\} \subset H^0\big(\mathrm{Sh}_K, F^0\big(\boldsymbol{V}^\otimes_{\mathrm{dR},E}\big)^{\nabla=0}\big).$$

Also, for any field $\kappa/\mathbb{Q}$ with algebraic closure $\overline{\kappa}$, and any point $x \in \mathrm{Sh}_K(\kappa)$, we obtain a canonical collection of $\mathrm{Gal}(\overline{\kappa}/\kappa)$-invariant tensors

$$\{s_{\alpha,\text{ét},x}\} \subset H^1_{\text{ét}}\big(\mathcal{A}_{x,\overline{\kappa}},\mathbb{Q}_p\big)^\otimes.$$

The point-wise stabilizer of this collection is isomorphic to $G_{\mathbb{Q}_p}$ and is therefore reductive.

Now, fix Φ,σ and $x_0 \in \mathbf{Z}_\Phi(\sigma)_{\mathscr{O}_{E,(v)}}(k)$ as above, and assume that $\operatorname{Spec} R_{\Phi,\sigma,x_0}$ intersects $\breve{\mathscr{S}}_K$ at x_0. Choose a finite extension L/E_v such that the generic fiber of every irreducible component of $\operatorname{Spec}(T_{\Phi,\sigma,x_0} \otimes_{\mathscr{O}_{E_v}} \mathscr{O}_L)$ is geometrically irreducible over L. Let L'/L be a finite extension such that there exists an L'-valued point $x : \operatorname{Spec} L' \to \breve{\mathscr{S}}_K \cap U^\circ_{\Phi,\sigma,x_0}$ specializing to x_0. Then we obtain a collection of Tate tensors $\{s_{\alpha,x_0}\} \subset \mathbb{D}(Q_0)(W_{\Phi,\sigma})$ via the process described in (3.3.4):

This uses the p-adic comparison isomorphism (4.4.3.2) (for a given choice of uniformizer $\pi \in \mathscr{O}_L$) and the Galois-invariant collection $\{s_{\alpha,\text{ét},x}\}$ contained in $H^1_{\text{ét}}\big(\mathcal{A}_{x,\overline{L}},\mathbb{Q}_p\big)^{\otimes}$.

Proposition 4.4.5. *Let $\mathcal{Z}$ be an irreducible component of* $\operatorname{Spec}(T_{\Phi,\sigma,x_0}\otimes_{\mathscr{O}_{E_v}}\mathscr{O}_L)$ *containing x, and let T be its ring of functions. Then:*

(1) As a quotient of $R_{\Phi,\sigma,x_0}\otimes\mathscr{O}_L$, T is adapted to $\{s_{\alpha,x_0}\}$ in the sense of (3.3.18).

(2) The collection $\{s_{\alpha,x_0}\}$ is independent of the lift x in $\mathrm{LM}(T)$ *and the choice of uniformizer π, and is therefore canonically attached to x_0 and the irreducible component* $\operatorname{Spec} T$.

Proof. We will use some notation from (4.3.5). Set $R = R_{\Phi,\sigma,x_0}$, $U^\circ = U^\circ_{\Phi,\sigma,x_0}$, $\widehat{U} = (\operatorname{Spf} R)^{\mathrm{an}}$ and let $\widehat{U}^\circ \subset \widehat{U}^{\mathrm{an}}$ be the complement of the boundary divisor. Recall from (3.2.13) that we have a canonical φ-equivariant horizontal isomorphism

$$\xi : \mathbb{D}(Q_0)(W_{\Phi,\sigma})\otimes_W R^{\mathrm{an,log}} \xrightarrow{\simeq} \mathbb{D}(\mathrm{Q}_{(\Phi,\sigma)})(R)\otimes_R R^{\mathrm{an,log}}$$

Therefore, the tensors $\{s_{\alpha,x_0}\}$ propagate to parallel, φ-invariant tensors $\{s^{\mathrm{an}}_\alpha = \xi(s_{\alpha,x_0})\}$ contained in $(\mathbb{D}(\mathrm{Q}_{(\Phi,\sigma)})(R)\otimes_R R^{\mathrm{an,log}})^{\otimes}$. By construction, (4.1.11) and the compatibility of the Hyodo-Kato isomorphism with ξ (3.2.15), the tensors $\{s^{\mathrm{an}}_\alpha\}$ are Hodge at x (cf. 3.3.8); in fact, they specialize to the tensors $\{s_{\alpha,\mathrm{dR},x}\}\subset H^1_{\mathrm{dR}}(\mathcal{A}_x)^{\otimes}$.

Also, if $(\mathcal{A},\lambda)$ is the polarized abelian variety over U° attached to $(\mathrm{Q}_{(\Phi,\sigma)},\lambda_{(\Phi,\sigma)})$, then we have (cf. 2.3.4) a natural identification of filtered coherent sheaves

$$\mathbb{D}(\mathrm{Q}_{(\Phi,\sigma)})|_{U^\circ_{\mathrm{Zar}}} = H^1_{\mathrm{dR}}(\mathcal{A}/U^\circ).$$

Here, U°_{Zar} is the Zariski site of U°. In particular, $\mathbb{D}(\mathrm{Q}_{(\Phi,\sigma)})|_{(\mathcal{Z}\cap U^\circ_{\mathscr{O}_L})_{\mathrm{Zar}}}$ is equipped with canonical horizontal Hodge tensors $\{s_{\alpha,\mathrm{dR}}\}$ that also specialize to $\{s_{\alpha,\mathrm{dR},x}\}$ at x.

Now, $\{s_{\alpha,\mathrm{dR}}\}$ and $\{s^{\mathrm{an}}_\alpha\}$ are horizontal tensors that agree at the point x of the smooth, geometrically irreducible space $\mathcal{Z}^{\mathrm{an}}\cap\widehat{U}^\circ_L$. So, for any $x'\in\mathrm{LM}(T)$ and any α, the specialization of s^{an}_α at x' is equal to $s_{\alpha,\mathrm{dR},x'}$. In particular, the collection $\{s^{\mathrm{an}}_\alpha\}$ is Hodge at every $x'\in\mathrm{LM}(T)$. Since the dimensional criterion in (3.3.17)(2) can be verified from the complex analytic uniformization of $\mathrm{Sh}_K(\mathbb{C})$, this finishes the proof of (1).

(2) follows from the argument in the proof of [Kis10, 2.3.5]. Suppose that $x'\in\mathrm{LM}(T)$ is another lift of x_0 and that $\{s_{\alpha,x_0'}\}$ is the resulting collection of Tate tensors (for some choice of uniformizer π'). Then, by (4.1.11), $\{s_{\alpha,\mathrm{dR},x'}\}$ is carried to $\{s_{\alpha,x_0'}\}$ under the Hyodo-Kato isomorphism. From the proof of (1), it now follows that $\{s_{\alpha,x_0'}\}=\{s_{\alpha,x_0}\}$. □

4.4.6. Choose any smooth admissible rppcd Σ' for $(\mathrm{GSp},\mathrm{S}^{\pm},K')$, and let Σ be the admissible rppcd for (G,X,K) induced from Σ' (cf. 4.2.20). Associated with this is the partial toroidal compactification $\mathscr{S}^{\Sigma'}_{K'}$ (cf. 4.3.7) of $\mathscr{S}_{K'}$: This is an integral model over $\mathscr{O}_{E,(v)}$ of the partial toroidal compactification $\mathrm{Sh}^{\Sigma'}_{K'}$ of $\mathrm{Sh}_{K'}$ from (4.2.19). Let $\mathbf{D}^{\Sigma'}_{K'}$ be the boundary divisor in $\mathscr{S}^{\Sigma'}_{K'}$, let $\check{\mathscr{S}}^{\Sigma}_K$ be the Zariski closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{S}^{\Sigma'}_{K'}$.

For every pair $(\Phi_G,\sigma_G)\in\mathbf{CLR}_K(G,X)$ with image (Φ,σ) in $\mathbf{CLR}_{K'}(\mathrm{GSp},\mathrm{S}^{\pm})$, let $\check{\mathbf{Z}}_{[(\Phi_G,\sigma_G)]}$ be the Zariski closure of $Z_{[(\Phi_G,\sigma_G)]}$ in $\check{\mathscr{S}}^{\Sigma}_K$. Also, let $Z_{\Phi_G}(\sigma_G)\subset\xi_{\Phi_G}(\sigma_G)$ (resp. $\mathbf{Z}_\Phi(\sigma)\subset\mathbf{\Xi}_\Phi(\sigma)$) be the closed stratum. Let $\check{\mathbf{Z}}_{\Phi_G}(\sigma_G)$ (resp. $\check{\mathbf{C}}_{\Phi_G}$, $\check{\mathscr{S}}_{K_{\Phi_G}}$) be the Zariski closure of $Z_{\Phi_G}(\sigma_G)$ (resp. C_{Φ_G}, $\mathrm{Sh}_{K_{\Phi_G}}$) in $\mathbf{Z}_\Phi(\sigma)_{\mathscr{O}_{E,(v)}}$ (resp. $\mathbf{C}_{\Phi,\mathscr{O}_{E,(v)}}$, $\mathscr{S}_{K'_\Phi,\mathscr{O}_{E,(v)}}$).

Theorem 4.4.7.

(1) The intersection $\check{\mathbf{D}}^{\Sigma}_K = \check{\mathscr{S}}^{\Sigma}_K\cap\mathbf{D}^{\Sigma'}_{K'}$ is a relative Cartier divisor over $\mathscr{O}_{E,(v)}$.

(2) We have:

$$\check{\mathscr{S}}^{\Sigma}_K = \bigcup_{[(\Phi_G,\sigma_G)]}\check{\mathbf{Z}}_{[(\Phi_G,\sigma_G)]},$$

where $[(\Phi_G, \sigma_G)]$ ranges over $\mathbf{Cusp}_K^{\Sigma}(G,X)$. This is compatible with the stratification of $\mathrm{Sh}_K^{\Sigma}(G,X)$ described in (4.2.19)(3).

(3) $\breve{\mathbf{Z}}_{[(\Phi_G,\sigma_G)]}$ is canonically isomorphic to $\breve{\mathbf{Z}}_{\Phi_G}(\sigma_G)$.

(4) $\breve{\mathscr{S}}_K$ is proper if and only if G is anisotropic modulo center.

Proof. (1) is a local statement. Let $x_0 \in \breve{\mathscr{S}}_K^{\Sigma}(k)$ be a closed point valued in a finite field k and lying in the $[(\Phi,\sigma)]$-stratum of $\mathscr{S}_{K'}^{\Sigma'}$. We can view x_0 as a point in $\mathbf{Z}_{\Phi}(\sigma)(k)$ and identify the complete local ring of $\mathscr{S}_{K'}^{\Sigma'}$ at x_0 with R_{Φ,σ,x_0}. Then, in the notation of (4.4.5), the complete local ring of $\breve{\mathscr{S}}_K^{\Sigma}$ at x_0 is identified with T_{Φ,σ,x_0}. Now, (4.4.5)(1), combined with (3.3.20)(2), shows that $\breve{\mathscr{S}}^{\Sigma} \cap \mathbf{D}_{K'}^{\Sigma'}$ is an effective relative Cartier divisor.

(2) and (3) are now immediate. We only need to observe that if Sh_K^{Σ} is the partial toroidal compactification of Sh_K attached to Σ, then the map $\mathrm{Sh}_K^{\Sigma} \to \mathrm{Sh}_{K'}^{\Sigma'}$ is simply the normalization of the Zariski closure of Sh_K in $\mathrm{Sh}_{K'}^{\Sigma'}$; cf. [Har89, 3.4].

By a result of Borel and Harish-Chandra [BHC62, 5.6] (cf. also [Pau04, 3.1.5]), G is anisotropic modulo center if and only if, for any level $K \subset G(\mathbb{A}_f)$, the Shimura variety $\mathrm{Sh}_K(G,X)$ is proper. Therefore, (4) is a consequence of (1): $\breve{\mathscr{S}}_K$ is proper precisely when $\mathscr{S}_K$ is proper, and the latter is proper precisely when the boundary $\breve{\mathbf{D}}_K^{\Sigma}$ is empty for any rppcd Σ. Since $\breve{\mathbf{D}}_K^{\Sigma}$ is flat over $\mathscr{O}_{E,(v)}$, it is empty if and only if its generic fiber is. □

4.4.8. As stated in the introduction, from (4.4.7), we can now easily deduce Morita's conjecture. Let us recall the **Mumford-Tate group** MT_A associated with an abelian variety A over $\mathbb{C}$: One way to define it is as the Tannaka group of the Tannakian sub-category of the category of polarizable rational Hodge structures generated by the rational Hodge structure $H^1(A(\mathbb{C}),\mathbb{Q})$ (cf. [DMOS82, §5]). In particular, it is a connected reductive group and there is a canonical map $h_A : \mathbb{S} \to \mathrm{MT}_{A,\mathbb{R}}$ that gives rise to the Hodge decomposition of $H^1(A(\mathbb{C}),\mathbb{C})$. The pair (MT_A, X_A), where X_A is the $\mathrm{MT}_A(\mathbb{R})$-conjugacy class of h_A, is a Shimura datum of Hodge type.

Suppose now that A is defined over a number field F. The **Mumford-Tate group** MT_A of A is MT_{σ^*A}, for any embedding $\sigma : F \hookrightarrow \mathbb{C}$. The main result of [DMOS82] shows that MT_A does not depend on the choice of embedding. The following theorem is originally due to Paugam-Vasiu-Lee [Pau04, Vas08, Lee12].

Theorem 4.4.9. *Suppose that MT_A is anisotropic modulo center. Then A has potentially good reduction at all finite places of F.*

Proof. Extending F if necessary, we can assume that it contains the reflex field $E = E(\mathrm{MT}_A, X_A)$. Fix $\sigma : F \hookrightarrow \mathbb{C}$, and set $V = H^1(\sigma^*A(\mathbb{C}),\mathbb{Q})$ equipped with a pairing attached to some polarization of σ^*A. We then have a natural embedding of Shimura data:

$$(\mathrm{MT}_A, X_A) \hookrightarrow (\mathrm{GSp}(V), \mathrm{S}^{\pm}).$$

Fix a prime p and a place $v|p$ for E. Choose $V_{\mathbb{Z}} \subset V$, $n \geq 3$ and a cone decomposition Σ' as in (4.4.1). Let $\mathscr{S}_{K'}^{\Sigma'}$ be the attached Chai-Faltings compactification over $\mathscr{O}_{E,(v)}$ as in *loc. cit.*.

With $K = K' \cap \mathrm{MT}_A(\mathbb{A}_f)$, we see from [Vas08, Fact 2.6] that, in order to show that A has potentially good reduction at any place $w|v$ of F, it is enough to show that the Zariski closure $\breve{\mathscr{S}}_K$ of $\mathrm{Sh}_K(\mathrm{MT}_A, X_A)$ in $\mathscr{S}_{K'}$ is proper. We have shown this already in (4.4.7)(4). □

Remark 4.4.10. We note that in [Vas12c, 2.2.6] Vasiu deduces the properness of $\breve{\mathscr{S}}_K$ from the validity of Morita's conjecture. Our arrow of deduction points in the opposite direction.

4.4.11. Let $\mathscr{S}_K$ be the normalization of $\breve{\mathscr{S}}_K$. If we have more information about the local structure of the special fiber of $\mathscr{S}_K$, we can improve on (4.4.7). Let (Φ_G, σ_G) be an object in $\mathbf{CLR}_K(G, X)$ with associated tuple (Φ, σ) in $\mathbf{CLR}_{K'}(\mathrm{GSp}, \mathrm{S}^{\pm})$. Let $\breve{\boldsymbol{\Xi}}_{\Phi_G}$ be the Zariski closure of ξ_{Φ_G} in $\boldsymbol{\Xi}_{\Phi,\mathscr{O}_{E,(v)}}$, and let $\boldsymbol{\Xi}_{\Phi_G}$ be its normalization. This gives us a tower:

$$\boldsymbol{\Xi}_{\Phi_G} \to \mathbf{C}_{\Phi_G} \to \mathscr{S}_{K_{\Phi_G}},$$

where $\mathbf{C}_{\Phi_G}$ (resp. $\mathscr{S}_{K_{\Phi_G}}$) is the normalization of $\breve{\mathbf{C}}_{\Phi_G}$ (resp. $\breve{\mathscr{S}}_{K_{\Phi_G}}$).

Lemma 4.4.12. *Let $k(v)$ be the residue field of E_v, and suppose that the special fiber $\mathscr{S}_{K,k(v)}$ is reduced. Choose an algebraic closure $\overline{k(v)}$ of $k(v)$, and let E_v^{nr} be a maximal unramified extension of E_v with residue field $\overline{k(v)}$. Then, for every point $x_0 \in \breve{\mathscr{S}}_K^{\Sigma}(\overline{k(v)})$, the generic fibers of the irreducible components of $\left(\breve{\mathscr{S}}_K^{\Sigma}\right)^{\wedge}_{x_0}$ are geometrically irreducible over E_v^{nr}.*

Proof. Let $\mathcal{Z}$ be an irreducible component of $\left(\breve{\mathscr{S}}_K^{\Sigma}\right)^{\wedge}_{x_0}$ and let T be its ring of functions. Let $\widetilde{\mathcal{Z}}$ be the normalization of $\mathcal{Z}$ and let $\widetilde{T}$ be its ring of functions. Note that $\widetilde{T}$ is again a local domain: this follows from [EGAIV2, 7.8.3(vii)].

Also by *loc. cit.*, $\widetilde{T}$ is the complete local ring for the normalization of $\breve{\mathscr{S}}_K^{\Sigma}$ at a point lying above x_0. Let $a \in T$ be the equation defining the boundary divisor, and let π_v be a uniformizer for E_v. By (4.4.7)(1), and our hypothesis on $\mathscr{S}_K$, we see that that $(\widetilde{T}/\pi_v\widetilde{T})[a^{-1}]$ is reduced and non-zero.

We find from [EGAIV2, 4.5.10] that $\mathcal{Z}_{\mathbb{Q}}$ is geometrically irreducible over E_v^{nr} precisely when E_v^{nr} is algebraically closed in the fraction field of T, which is of course also the fraction field of $\widetilde{T}[a^{-1}]$.

So it now suffices to prove the following claim: If R is a faithfully flat normal domain over $\mathscr{O}_{E_v^{\mathrm{nr}}}$ such that $R/\pi_v R$ is reduced, then E_v^{nr} is algebraically closed in the fraction field of R. Indeed, localizing at a minimal prime over $\pi_v R$, we can assume that R is also a discrete valuation ring, and our assumption shows that R is unramified over $\mathscr{O}_{E_v^{\mathrm{nr}}}$. ☐

Theorem 4.4.13. *Suppose that the special fiber $\mathscr{S}_{K,k(v)}$ is* reduced. *Let $\mathscr{S}_K^{\Sigma}$ be the normalization of $\breve{\mathscr{S}}_K^{\Sigma}$. Then:*

1. *$\mathscr{S}_K^{\Sigma}$ admits a stratification by normal, $\mathscr{O}_{E,(v)}$-flat sub-schemes:*

$$\mathscr{S}_K^{\Sigma} = \bigsqcup_{[(\Phi_G,\sigma_G)]} \mathbf{Z}_{[(\Phi_G,\sigma_G)]},$$

where $[(\Phi_G, \sigma_G)]$ ranges over $\mathbf{Cusp}_K^{\Sigma}(G, X)$. This is compatible with the stratification of $\mathrm{Sh}_K^{\Sigma}(G, X)$ described in (4.2.19)(3).

2. *For every Φ_G in $\mathbf{CLR}_K(G, X)$, $\boldsymbol{\Xi}_{\Phi_G}$ is an $\mathbf{E}_{\Phi_G}$-torsor over $\mathbf{C}_{\Phi_G}$.*

3. *$\mathbf{Z}_{[(\Phi_G,\sigma_G)]}$ is canonically isomorphic to the closed stratum $\mathbf{Z}_{\Phi_G}(\sigma_G)$ in the twisted torus embedding $\boldsymbol{\Xi}_{\Phi_G}(\sigma_G)$ attached to σ_G. Moreover, the completion $\mathfrak{X}_{[(\Phi_G,\sigma_G)]}$ of $\mathscr{S}_K^{\Sigma}$ along $\mathbf{Z}_{[(\Phi_G,\sigma_G)]}$ is compatibly isomorphic to the completion of $\boldsymbol{\Xi}_{\Phi_G}(\sigma_G)$ along $\mathbf{Z}_{\Phi_G}(\sigma_G)$.*

Proof. Let $x_0 \in \breve{\mathscr{S}}_K^{\Sigma}(\overline{k(v)})$ be a closed point lying in the $[(\Phi, \sigma)]$-stratum of $\mathscr{S}_{K'}^{\Sigma'}$. Identify the complete local ring of $\mathscr{S}_{K'}^{\Sigma'}$ at x_0 with R_{Φ,σ,x_0}, as in (4.4.5). Let T be a quotient domain of T_{Φ,σ,x_0} such that $\operatorname{Spec} T \subset \operatorname{Spec} T_{\Phi,\sigma,x_0}$ is an irreducible component. Then (4.4.5), along with (4.4.12), shows that there exists a canonical collection of Tate tensors $\{s_{\alpha,x_0}\}$ such that T is adapted to $\{s_{\alpha,x_0}\}$. Now, (3.3.20) shows that the $\mathbf{E}_{\Phi}$-torsor $\boldsymbol{\Xi}_{\Phi,T^{\mathrm{sab}}}$ has a canonical reduction of structure group to an $\mathbf{E}_{\Phi_G}$-torsor $\boldsymbol{\Xi}_{\Phi_G,T^{\mathrm{sab}}}$. Moreover, we find by *loc. cit.* that, if $T_{\mathrm{n}}^{\mathrm{sab}}$ is the normalization of T^{sab}, then the normalization T_{n} of T can be identified with the complete local ring at x_0 of the torus embedding $\boldsymbol{\Xi}_{\Phi_G,T_{\mathrm{n}}^{\mathrm{sab}}} \hookrightarrow \boldsymbol{\Xi}_{\Phi_G,T_{\mathrm{n}}^{\mathrm{sab}}}(\sigma_G)$.

Let $\mathbf{Z}_{[(\Phi_G,\sigma_G)]}$ be the Zariski closure of $Z_{[(\Phi_G,\sigma_G)]}$ in $\mathscr{S}^{\Sigma}_K$. Then x_0, viewed as a point $(\operatorname{Spec} T_{\mathrm{n}})(\overline{k(v)})$, belongs to $\mathbf{Z}_{[(\Phi_G,\sigma_G)]}(\overline{k(v)})$. Moreover, the complete local ring $\mathbf{Z}_{[(\Phi_G,\sigma_G)]}$ at x_0 can be identified with that of the closed stratum in $\Xi_{\Phi_G,T^{\mathrm{sab}}_{\mathrm{n}}}(\sigma_G)$. It is now easy to deduce all the numbered assertions of the theorem from (4.4.7) and the corresponding statements in characteristic 0 (cf. 4.2.19) □

4.5. **Compactifying $\mathbf{M}_{V_{\mathbb{Z}},n,\psi,\mathbb{Z}_{(p)}}$ when $p^2\nmid d$.** We fix a symplectic space (V,ψ) over $\mathbb{Q}$ and a polarized $\mathbb{Z}$-lattice $V_{\mathbb{Z}}\subset V$ of discriminant d^2. We will assume that $g=\dim V\geq 2$. Set

$$r = 2^{\epsilon}\prod_{\substack{\text{primes } \ell>2\\ \ell^2|d}}\ell,$$

where $\epsilon=1$ if $2\mid d$ and $\epsilon=0$, otherwise. Given an integer $n\geq 3$, our goal here is to show that the results of Chai-Faltings-Lan presented in (4.3.7), which have to do with good compactifications of the restriction of $\mathbf{M}_{V_{\mathbb{Z}},n,\psi}$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$, can be extended over $\mathbb{Z}\left[\frac{1}{nr}\right]$. Maintaining the notation from (4.3.2), we will write $K(n)\subset \mathrm{GSp}(V)(\mathbb{A}_f)$ for the level n subgroup associated with $V_{\mathbb{Z}}$, and $\mathscr{S}_{K(n)}$ for the pull-back of $\mathbf{M}_{V_{\mathbb{Z}},n,\psi}$ over $\mathbb{Z}\left[\frac{1}{nr}\right]$. The generic fiber of $\mathscr{S}_{K(n)}$ can of course be identified with $\mathrm{Sh}_{K(n)}(\mathrm{GSp}(V),\mathrm{S}^{\pm})$.

4.5.1. Fix a prime p such that $p^2\nmid d$. For the sake of completeness, we will now briefly describe the local structure of $\mathscr{S}_{K(n),\mathbb{Z}_{(p)}}$ for $p\nmid n$, following [RZ96] and [Gör03]. We also take the opportunity to show how the powerful results of Vasiu-Zink [VZ10] can be applied to define integral canonical models of Shimura varieties, and to show their existence, even at primes of bad reduction; cf. [MP12a] for a similar application to orthogonal Shimura varieties.

Let $\mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}$ be the projective $\mathbb{Z}_{(p)}$-scheme such that, for every $\mathbb{Z}_{(p)}$-algebra R, we have:

$$\mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}(R) = \begin{pmatrix}\text{Isotropic } R\text{-sub-modules } \mathrm{Fil}^1 V_R\subset V_R\\ \text{that are, locally on } \operatorname{Spec} R, \text{ direct summands of rank } g.\end{pmatrix}$$

Let $\mathrm{rad}(V_{\mathbb{F}_p})\subset V_{\mathbb{F}_p}$ be the radical for the induced alternating form on $V_{\mathbb{F}_p}$: this is trivial if $p\nmid d$, and is a two-dimensional sub-space if $p\mid d$. For any point $x_0\in \mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}(\overline{\mathbb{F}}_p)$ let $\mathrm{Fil}^1_{x_0}\subset V_{\overline{\mathbb{F}}_p}$ denote the attached isotropic sub-space. The proof of the following proposition can be gleaned from [Gör03, §5.1]

Proposition 4.5.2.

(1) *If* $\mathrm{Fil}^1_{x_0}$ *does not contain* $\mathrm{rad}(V_{\mathbb{F}_p})$*, then* $\mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}$ *is smooth at* x_0.

(2) *If* $\mathrm{Fil}^1_{x_0}$ *contains* $\mathrm{rad}(V_{\mathbb{F}_p})$*, then there exists a neighborhood* U *of* x_0 *and an isomorphism of* $\mathbb{Z}_{(p)}$*-schemes*

$$U\xrightarrow{\simeq}\mathbb{A}^{\frac{g(g+1)}{2}-3}_{\mathbb{Z}_{(p)}}\times_{\operatorname{Spec}\mathbb{Z}_{(p)}}\operatorname{Spec}\mathbb{Z}_{(p)}\frac{[T_1,T_2,T_3,T_4]}{(T_1T_4-T_2T_3-p)}.$$

□

We now recall some definitions from [VZ10].

Definition 4.5.3. A $\mathbb{Z}_{(p)}$-scheme X is **healthy regular** if it is regular, faithfully flat over $\mathbb{Z}_{(p)}$, and if, for every open sub-scheme $U\subset X$ containing $X_{\mathbb{Q}}$ and all generic points of $X_{\mathbb{F}_p}$, every abelian scheme over U extends uniquely to an abelian scheme over X.

A local $\mathbb{Z}_{(p)}$-algebra R with maximal ideal $\mathfrak{m}$ is **quasi-healthy regular** if it is regular, faithfully flat over $\mathbb{Z}_{(p)}$, and if every abelian scheme over Spec $R\backslash\{\mathfrak{m}\}$ extends uniquely to an abelian scheme over Spec R.

Theorem 4.5.4 (Vasiu-Zink)**.** *Let R be a regular local $\mathbb{Z}_{(p)}$-algebra with algebraically closed residue field k, of dimension at least 2. Suppose that it admits a surjection*

$$R \twoheadrightarrow W(k)[|T_1,T_2|]/(p-h),$$

where $h \notin (p, T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$. Then R is quasi-healthy regular. In particular, if R is a formally smooth $\mathbb{Z}_{(p)}$-algebra of dimension at least 2, then R is quasi-healthy regular.

Proof. This is [VZ10, Theorem 3]. □

Corollary 4.5.5. *Suppose either that $p > 2$ or that $g > 2$. Then $\mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}$ is healthy regular.*

Proof. This follows from the local description of $\mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}$ in (4.5.2) and the criterion from (4.5.4). □

Definition 4.5.6. A pro-scheme X over $\mathbb{Z}_{(p)}$ (for some prime p) satisfies the **extension property** if, for any healthy regular $\mathbb{Z}_{(p)}$-scheme S, any map $S_{\mathbb{Q}} \to X$ extends to a map $S \to X$.

A pro-scheme X over $\mathbb{Z}_{(p)}$ is an **integral canonical model** of its generic fiber $X_{\mathbb{Q}}$ if it is healthy regular and has the extension property. Clearly, if X is an integral canonical model of $X_{\mathbb{Q}}$, then it is uniquely determined by this property.

Consider the inverse system $\{\mathscr{S}_{K(n),\mathbb{Z}_{(p)}}\}_{(n,p)=1}$, where the set of integers n satisfying $(n,p) = 1$ is ordered by divisibility. The transition map from $\mathscr{S}_{K(n),\mathbb{Z}_{(p)}}$ to $\mathscr{S}_{K(m),\mathbb{Z}_{(p)}}$ for $m \mid n$ is the finite étale map that extracts a level-m structure from a level-n structure in the obvious way.

Proposition 4.5.7. *Suppose that $p > 2$. Then the pro-$\mathbb{Z}_{(p)}$-scheme*

$$\varprojlim_{(n,p)=1} \mathscr{S}_{K(n),\mathbb{Z}_{(p)}}.$$

is the integral canonical model of its generic fiber.

Proof. That this pro-scheme has the extension property follows from the Nerón-Ogg-Shafarevich criterion for good reduction of abelian varieties over local fields, and the definition of healthy regularity; cf. [Kis10, 2.3.8]. So we only have to check that it is itself healthy regular. When $p > 2$, $\mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}$ is a *local model* for $\mathscr{S}_{K(n),\mathbb{Z}_{(p)}}$ in the sense of [RZ96, DP94]; cf. [Pap00, Theorem 2.2]. This means that there exists an étale covering $V \to \mathscr{S}_{K(n)}$ equipped with an étale map $V \to \mathrm{M}^{\mathrm{loc}}_{\mathbb{Z}_{(p)}}$. So the proposition follows from (4.5.7). □

Definition 4.5.8. Mildly abusing terminology, we will also refer to $\mathscr{S}_{K(n),\mathbb{Z}_{(p)}}$ as the **integral canonical model** of its generic fiber; cf. also (4.6.6) below.

4.5.9. We will now review Zarhin's trick (cf. [Zar77, §2] or [Zar85, §4]), and formulate it in both moduli and group theoretic terms. To begin, fix a quadruple of integers x, y, z, w such that $x^2 + y^2 + z^2 + w^2 = d^2 - 1$. Set

$$\beta = \begin{pmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{pmatrix} \in \mathrm{End}(\mathbb{Z}^4).$$

If $\beta^\top$ is the transpose matrix, then we have $\beta^\top\beta = (d^2-1)I_4$, where I_4 is the identity matrix.

Set $V' = V^4 \oplus (V^\vee)^4$; we will equip it with a symplectic pairing ψ' defined for $w_1, w_2 \in V^4$ and $f_1, f_2 \in (V^\vee)^4$ by the following formula:

$$\psi'\big((w_1,f_1),(w_2,f_2)\big) = \psi^4(w_1,w_2) + d^2(\psi^\vee)^4(f_1,f_2) - f_2(\beta^\top(w_1)) + f_1(\beta^\top(w_2)).$$

In this formula, $\psi^\vee$ is the dual symplectic pairing on $V^\vee$ attached to ψ; ψ^4 and $(\psi^\vee)^4$ are the induced pairings on V^4 and $(V^\vee)^4$, respectively; and $\beta^\top$ is being viewed as an endomorphism

of V^4 via the identification $V^4 = V \otimes_{\mathbb{Z}} \mathbb{Z}^4$. The key point now is that ψ' restricts to a perfect symplectic pairing on $V'_{\mathbb{Z}}$ (this is a routine check, using the identity $\alpha^\top \alpha = d^2 - 1$).

The natural action of $\mathrm{GSp} \coloneqq \mathrm{GSp}(V,\psi)$ on V' embeds it within $\mathrm{GSp}' \coloneqq \mathrm{GSp}(V',\psi')$, and gives rise to an embedding of Shimura data

$$(\mathrm{GSp}, \mathrm{S}^{\pm}(V)) \hookrightarrow (\mathrm{GSp}', \mathrm{S}^{\pm}(V')).$$

For any $n \in \mathbb{Z}_{>0}$, let $K'(n) \subset \mathrm{GSp}'(\mathbb{A}_f)$ be the level n sub-group attached to $V'_{\mathbb{Z}} = V^4_{\mathbb{Z}} \oplus (V^\vee_{\mathbb{Z}})^4$. Then we obtain maps of Shimura varieties:

$$\mathfrak{z}_{\beta,n} : \mathrm{Sh}_{K(n)}(\mathrm{GSp}, \mathrm{S}^{\pm}(V)) \to \mathrm{Sh}_{K'(n)}(\mathrm{GSp}, \mathrm{S}^{\pm}(V')).$$

Viewed as a map of moduli spaces

$$\mathfrak{z}_{\beta,n} : \mathscr{S}_{K(n),\mathbb{Q}} = \mathbf{M}_{V_{\mathbb{Z}},n,\psi,\mathbb{Q}} \to \mathbf{M}_{V'_{\mathbb{Z}},n,\psi',\mathbb{Q}} = \mathscr{S}_{K'(n),\mathbb{Q}},$$

this can be described as follows: Given a tuple (A,λ,ν,α) on the left hand side, we map it to the tuple $(A',\lambda',\nu,\alpha')$, where $A' = A^4 \times (A^\vee)^4$, $\alpha' = \alpha^4 \times (\lambda \circ \alpha)^4$, and $\lambda' : A' \to (A')^\vee$ is the unique principal polarization making the following diagram commute:

$$\begin{tikzcd}
A^4 \times A^4 \arrow[rr, "f = \begin{pmatrix} 1 & [\beta] \\ 0 & \lambda^4 \end{pmatrix}"] \arrow[d, "\lambda^4 \times \lambda^4"'] & & A^4 \times (A^\vee)^4 = A' \arrow[d, "\lambda'"] \\
(A^\vee)^4 \times (A^\vee)^4 & & (A^\vee)^4 \times A^4 = (A')^\vee . \arrow[ll, "f^\vee"]
\end{tikzcd}$$

Here, $[\beta] : A^4 \to A^4$ is the natural map determined by β via the identification $A^4 = A \otimes_{\mathbb{Z}} \mathbb{Z}^4$.

This moduli theoretic description makes sense over any base, and so we obtain an extension

$$\mathfrak{z}_{\beta,n,\mathbb{Z}[\frac{1}{nr}]} : \mathscr{S}_{K(n)} \to \mathscr{S}_{K'(n),\mathbb{Z}[\frac{1}{nr}]}.$$

Lemma 4.5.10. *$\mathfrak{z}_{\beta,n,\mathbb{Z}[\frac{1}{nr}]}$ is a finite map. In particular, $\mathscr{S}_{K(n)}$ is the normalization of the Zariski closure of the image (under $\mathfrak{z}_{\beta,n}$) of $\mathrm{Sh}_{K(n)}(\mathrm{GSp}, \mathrm{S}^{\pm}(V))$ in $\mathscr{S}_{K'(n),\mathbb{Z}[\frac{1}{nr}]}$.*

Proof. It is easy to see, using the Nerón-Ogg-Shafarevich criterion, that the map is proper. We have to check that it is quasi-finite: for this, it is enough to show that, given a polarized abelian variety (A,λ) over an algebraically closed field k, there are, up to isomorphism, only finitely many polarized abelian varieties (B,μ) over k with $(A^8,\lambda^8) \simeq (B^8,\mu^8)$. This follows from [Zar77, 4.2.2]. □

4.5.11. Choose n so large that $\mathfrak{z}_{\beta,n}$ is a closed immersion. Suppose that we have $\Phi \in \mathbf{CLR}_{K(n)}(\mathrm{GSp}, \mathrm{S}^{\pm}(V))$, with associated $\Phi' \in \mathbf{CLR}_{K'(n)}(\mathrm{GSp}', \mathrm{S}^{\pm}(V'))$ (cf. 4.2.11). Taking the normalization of the Zariski closure of the tower $\xi_\Phi \to C_\Phi \to \mathrm{Sh}_{K_\Phi}$ in the tower $\mathbf{\Xi}_{\Phi'} \to \mathbf{C}_{\Phi'} \to \mathscr{S}_{K_{\Phi'}}$ gives us (cf. 4.4.11):

$$\mathbf{\Xi}_\Phi \to \mathbf{C}_\Phi \to \mathscr{S}_{K_\Phi}. \tag{4.5.11.1}$$

On the other hand, applying the method of (4.3.3) to Φ, we can attach to it a tuple $(W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi_1, \delta)$ (cf. 3.1.7), and construct the associated tower as in (3.1.13) (we consider this again over $\mathbb{Z}\left[\frac{1}{nr}\right]$):

$$\mathbf{\Xi}_{\Phi_1,\delta} \to \mathbf{C}_{\Phi_1,\delta} \to \mathbf{M}_{W_\bullet}. \tag{4.5.11.2}$$

Lemma 4.5.12. *The tower (4.5.11.2) is canonically isomorphic to the tower (4.5.11.1). In particular, $\mathscr{S}_{K_\Phi}$ is healthy regular with singularities as described in (4.5.2), $\mathbf{C}_\Phi$ is an abelian scheme over $\mathscr{S}_{K_\Phi}$, and $\mathbf{\Xi}_\Phi$ is smooth over $\mathscr{S}_{K_\Phi}$.*

Proof. As in the proof of (4.3.3), the generic fibers of the two towers are canonically isomorphic. Let $(W_\bullet V'_{\mathbb{Z}/n\mathbb{Z}}, \Phi'_1, \delta')$ be the tuple attached to Φ' by the method of (4.3.3). By definition, the tower $\mathbf{\Xi}_{\Phi'} \to \mathbf{C}_{\Phi'} \to \mathscr{S}_{K_{\Phi'}}$ attached to Φ' is identified with the tower

$$\mathbf{\Xi}_{\Phi'_1,\delta'} \to \mathbf{C}_{\Phi'_1,\delta} \to \mathbf{M}_{W_\bullet V'_{\mathbb{Z}/n\mathbb{Z}}}. \tag{4.5.12.1}$$

To show the lemma, it is enough to show that there is a natural finite map from the tower in (4.5.11.2) to that in (4.5.12.1) extending that on the generic fibers. This can be checked using (4.5.10). □

We can now improve (4.3.7) as follows:

Theorem 4.5.13. *Fix $n \geq 3$ and let $K = K(n)$. There exists a co-final system $\{\Sigma\}$ of rppcds for $(\mathrm{GSp}, \mathrm{S}^\pm, K)$, such that, for each Σ in this system, there exists a regular algebraic space $\mathscr{S}_K^\Sigma$ over $\mathbb{Z}\left[\frac{1}{nr}\right]$ containing $\mathscr{S}_K$ as a open dense sub-scheme and satisfying the following properties:*

1. *The fiber of $\mathscr{S}_K^\Sigma$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$ is identified with the Chai-Faltings compactification from (4.3.7).*
2. *The boundary $\mathbf{D}_K^\Sigma = \mathscr{S}_K^\Sigma \backslash \mathscr{S}_K$ is a relative effective Cartier divisor.*
3. *If Σ is complete, then $\mathscr{S}_K^\Sigma$ is proper over $\mathbb{Z}\left[\frac{1}{nr}\right]$.*
4. *$\mathscr{S}_K^\Sigma$ admits a stratification by (healthy) regular sub-schemes:*
$$\mathscr{S}_K^\Sigma = \bigsqcup_{[(\Phi,\sigma)]} \mathbf{Z}_{[(\Phi,\sigma)]},$$
where $[(\Phi,\sigma)]$ ranges over $\mathbf{Cusp}_K^\Sigma(\mathrm{GSp}, \mathrm{S}^\pm)$. This is compatible with the stratification of its generic fiber in (4.2.19)(3).
5. *For every $[(\Phi,\sigma)] \in \mathbf{Cusp}_K^\Sigma(\mathrm{GSp}, \mathrm{S}^\pm)$ with representative (Φ,σ), $\mathbf{Z}_{[(\Phi,\sigma)]}$ is canonically isomorphic to the closed stratum $\mathbf{Z}_\Phi(\sigma)$ in the twisted torus embedding $\mathbf{\Xi}_\Phi(\sigma)$. In fact, the completion $\mathfrak{X}_{[(\Phi,\sigma)]}$ of $\mathscr{S}_K^\Sigma$ along $\mathbf{Z}_{[(\Phi,\sigma)]}$ is canonically isomorphic to the completion of $\mathbf{\Xi}_\Phi(\sigma)$ along $\mathbf{Z}_\Phi(\sigma)$.*
6. *Every complete Σ admits a refinement Σ' such that $\mathscr{S}_K^{\Sigma'}$ is projective.*

Proof. This is immediate from (4.4.7) and (4.4.13). □

Remark 4.5.14. Even though the Siegel Shimura variety is of PEL type, the embedding arising from Zarhin's trick realizes GSp as a sub-group in GSp$'$ in a somewhat subtle way, involving tensors that are not just those generated by endomorphisms of V'. So, even in this special case, one really needs to be able to work with arbitrary Hodge tensors to show transversality of the intersection with the boundary.

Remark 4.5.15. Along the lines of [FC90, IV.5.7(5)] or [Lan08, 6.4.1.1(6)], we can now check that the universal polarized abelian scheme over $\mathscr{S}_{K(n)}$ extends to a semi-abelian scheme $\mathrm{G} \to \mathscr{S}_{K(n)}^\Sigma$, and to give a criterion for a semi-abelian scheme $G \to T$ over a noetherian normal $\mathbb{Z}\left[\frac{1}{nr}\right]$-scheme T to arise as the pull-back of a map $T \to \mathscr{S}_{K(n)}^\Sigma$. This is because the completions along the various strata come equipped with canonical polarized log 1-motifs, which in turn give rise to semi-abelian schemes using (2.3.4). These can then be algebraized over an étale neighborhood of the strata as in [FC90, §IV.4].

Remark 4.5.16. The same method allows one to construct and describe good compactifications of integral models of many PEL Shimura varieties with parahoric level at p. There are essentially two requirements: Reducedness of the special fiber, which is known in many cases; cf. [Gör03, Gör01, Pap00, PZ12] and having an analogue of the explicit description (as in (4.5.12)) of the tower (4.5.11.1).

4.6. **Smooth compactifications of Hodge type.** Let us start with a Shimura datum (G, X), and a rational prime p. Suppose that G is **unramified at** p: this means that $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified extension. This is also equivalent to saying that G has a reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$.

Definition 4.6.1. An embedding $\iota : (G, X) \hookrightarrow (G', X')$ of Shimura data is said to be **p-integral** if there exist reductive models $G_{\mathbb{Z}_{(p)}}$ of G and $G'_{\mathbb{Z}_{(p)}}$ of G' over $\mathbb{Z}_{(p)}$, and if the embedding of groups $G \hookrightarrow G'$ underlying i is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow G'_{\mathbb{Z}_{(p)}}$.

Lemma 4.6.2. *Suppose that (G, X) is a Shimura datum of Hodge type such that G is unramified at p with reductive model $G_{\mathbb{Z}_{(p)}}$.*

1. *There exists a p-integral embedding of Shimura data $\iota : (G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$ into a Siegel Shimura datum.*
2. *Suppose that the embedding $G \hookrightarrow \mathrm{GSp}$ arises from an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSp}_{\mathbb{Z}_{(p)}} = \mathrm{GSp}(V_{\mathbb{Z}_{(p)}}, \psi)$, for a symplectic $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$. Then there exists a collection of tensors $\{s_\alpha\} \subset V^{\otimes}_{\mathbb{Z}_{(p)}}$ such that $G_{\mathbb{Z}_{(p)}} \subset \mathrm{GSp}_{\mathbb{Z}_{(p)}}$ is the pointwise stabilizer of $\{s_\alpha\}$.*

Proof. For (1), choose any embedding $\iota' : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), \mathrm{S}^{\pm})$ of Shimura data. By [Kis10, 2.3.1], there exist a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$ and an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ that induces ι' over $\mathbb{Q}$. The problem is that ψ might not induce a perfect $\mathbb{Z}_{(p)}$-pairing on $V_{\mathbb{Z}_{(p)}}$. To take care of this, we apply Zarhin's trick (4.5.9), which tells us that there exists a perfect pairing ψ' on $V'_{\mathbb{Z}_{(p)}} = \big(V_{\mathbb{Z}_{(p)}} \times V^{\vee}_{\mathbb{Z}_{(p)}}\big)^4$ and an embedding $\mathrm{GSp}(V, \psi) \hookrightarrow \mathrm{GSp}(V', \psi')$. This also induces an embedding of the corresponding Shimura data. We can then check that the induced embedding $(G, X) \hookrightarrow (\mathrm{GSp}(V', \psi'), S^{\pm})$ arises from an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSp}(V'_{\mathbb{Z}_{(p)}}, \psi')$ and is thus p-integral.

We note that, in [Kis10, 2.3.1], when $p = 2$, Kisin restricts attention to groups G without a factor of type B. In fact, this restriction, which arises from a corresponding restriction in a result of G. Prasad and J.-K. Yu, is unnecessary. To be more precise, consider the following assertion:

- Let $i : \mathscr{G} \to \mathscr{H}$ be a map of reductive group schemes over $\mathbb{Z}_p$, whose fiber over $\mathbb{Q}_p$ is a closed embedding; then i is a closed embedding.

According to [PY06, 1.3], this assertion is true as long as $\mathscr{G}_{\overline{\mathbb{Q}}_p}$ does not admit a normal sub-group isomorphic to SO_{2n+1}. In [Kis10, 2.3.1], this assertion needs to be applied when $\mathscr{G} = G_{\mathbb{Z}_p}$. But the classification of Shimura data of Hodge type in [Del79, 1.3] shows that $G_{\overline{\mathbb{Q}}}$ can never have a normal sub-group isomorphic to SO_{2n+1}. Indeed, any factor of the derived sub-group G^{der} of type B will be simply connected.

(2) follows from [Kis10, 1.3.2]. □

4.6.3. We will now fix an unramified-at-p Shimura datum (G, X), a reductive model $G_{\mathbb{Z}_{(p)}}$ for G over $\mathbb{Z}_{(p)}$, and a p-integral embedding $(G, X) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm})$. Let (V, ψ) be the symplectic space to which GSp is attached, and let $V_{\mathbb{Z}_{(p)}} \subset V$ be the $\mathbb{Z}_{(p)}$-lattice on which ψ such that $G \hookrightarrow \mathrm{GSp}$ is obtained from an embedding of reductive $\mathbb{Z}_{(p)}$-groups $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSp}\big(V_{\mathbb{Z}_{(p)}}\big)$. Fix a polarized $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ such that $V_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)} = V_{\mathbb{Z}_{(p)}}$, and of discriminant d^2. For every integer n such that $(n, p) = 1$, let $K'(n) \subset \mathrm{GSp}(\mathbb{A}_f)$ be the sub-group of level n associated with $V_{\mathbb{Z}}$, and let $K(n) = K'(n) \cap G(\mathbb{A}_f)$. Set $K'_p = \mathrm{GSp}(\mathbb{Z}_p)$, $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$.

For n large enough, it follows from (4.1.5) that we have a closed embedding of Shimura varieties $\mathrm{Sh}_{K(n)}(G, X) \hookrightarrow \mathrm{Sh}_{K'(n)}(\mathrm{GSp}, \mathrm{S}^{\pm})_E$ over the reflex field $E = E(G, X)$. Note that $\mathrm{Sh}_{K'(n)}(\mathrm{GSp}, \mathrm{S}^{\pm})$ is just the fine moduli space $\mathbf{M}_{V_{\mathbb{Z}/n\mathbb{Z}}, \psi} \otimes_{\mathbb{Z}[1/n]} \mathbb{Q}$ introduced in (3.1.1), and as such has the integral canonical model $\mathscr{S}_{K'(n)}(\mathrm{GSp}, \mathrm{S}^{\pm}) = \mathbf{M}_{V_{\mathbb{Z}/n\mathbb{Z}}, \psi}$ over $\mathbb{Z}\left[\frac{1}{nd}\right]$.

The reflex field E is unramified at p (cf. [Mil92, 4.4.7]). Let $v|p$ be a prime of E lying above p, and let $\mathscr{O}_{E,(v)}$ be the localization of $\mathscr{O}_E$ at v. Let $\mathscr{S}_{K(n)}(G,X)$ be the normalization of the Zariski closure of $\mathrm{Sh}_{K(n)}(G,X)$ in $\mathscr{S}_{K'(n)}(\mathrm{GSp},\mathrm{S}^{\pm})_{\mathscr{O}_{E,(v)}}$.

Theorem 4.6.4 (Kisin, Vasiu)**.** *Suppose that $p>2$. Set*

$$\mathscr{S}_{K_p}(G,X) = \varprojlim_{(n,p)=1} \mathscr{S}_{K(n)}(G,X).$$

Then:

(1) For each n, $\mathscr{S}_{K(n)}(G,X)$ is smooth over $\mathscr{O}_{E,(v)}$, and the transition maps in the inverse limit above are finite étale.

(2) $\mathscr{S}_{K_p}(G,X)$ satisfies the extension property (4.5.6) and is therefore the integral canonical model for $\mathrm{Sh}_{K_p}(G,X)$.

Proof. This is [Kis10, 2.3.8]. □

Remark 4.6.5. In the case $p=2$, work of Vasiu-Zink [VZ10] reduces the problem again to showing the smoothness of $\mathscr{S}_{K(n)}(G,X)$. It should be possible to prove smoothness using the strategy in [Kis10]. There are three results in *loc. cit.* whose proofs use the condition $p>2$. The first is in [Kis10, 2.3.1], where, as we observed in the proof of (4.6.2), the condition is not in fact needed. The second is [Kis10, 1.4.2], which has since been extended to the case $p=2$ by the work of W. Kim [Kim11], Lau [Lau12], and T. Liu [Liu11]. Finally, the assumption on p is also used in a deformation theoretic argument in [Kis10, 1.5.8]. When the universal abelian scheme over $\mathscr{S}_{K(n)}(G,X)$ has connected p-divisible group at every closed point, Zink's theory of displays allows Kisin to push the argument through even when $p=2$. In general, the condition on p does seem a little more serious here, and we have been unable to remove it so far.

Nonetheless, Vasiu has made some progress towards integral canonical models even when $p=2$; cf. [Vas12a, Vas12b, Vas12d], and especially [Vas12c], where the case $p=2$ is claimed to have been tackled in a number of cases, including those where the ordinary locus is dense in the special fiber of $\mathscr{S}_{K(n)}(G,X)$.

Definition 4.6.6. For any neat compact open sub-group $K^p\subset G(\mathbb{A}_f^p)$, we will refer to the quotient $\mathscr{S}_{K_pK^p}(G,X) := \mathscr{S}_{K_p}(G,X)/K^p$ as the **integral canonical model** for $\mathrm{Sh}_{K_pK^p}(G,X)$ over $\mathscr{O}_{E,(v)}$. In particular, for $n\geq 3$, the $\mathscr{O}_{E,(v)}$-scheme denoted $\mathscr{S}_{K(n)}(G,X)$ above is the integral canonical model for $\mathrm{Sh}_{K(n)}(G,X)$.

Definition 4.6.7. We will say that a triple (G,X,K) consisting of a Shimura datum (G,X) and a compact open $K\subset G(\mathbb{A}_f)$ is **unramified at** p if $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, for some reductive model $G_{\mathbb{Z}_{(p)}}$ of G over $\mathbb{Z}_{(p)}$.

If $K\subset G(\mathbb{A}_f)$ and $K'\subset G'(\mathbb{A}_f)$ are compact open sub-groups such that (G,X,K) and (G',X',K') are unramified at p, then a p**-integral embedding** $(\iota,g):(G,X,K)\hookrightarrow(G',X',K')$ consists of a p-integral embedding $\iota:(G,X)\hookrightarrow(G',X')$, and an element $g\in G'(\mathbb{A}_f)$ with $g_p\in K'_p$, such that $g^{-1}\iota(K)g\subset K'$. Here g_p is the p-primary part of g.

Suppose that we are given an unramified-at-p triple (G,X,K) corresponding to a reductive model $G_{\mathbb{Z}_{(p)}}$ for G.

Definition 4.6.8. A p**-integral CLR** for (G,X,K) is a CLR in $\mathbf{CLR}_K(G,X)$ of the form (P,X^+,g), with $g_p\in K_p$; here, g_p is the p-primary part of g. If Φ,Φ' are two p-integral CLRs then a map $\Phi\xrightarrow{\gamma}\Phi'$ between them is a map of CLRs with $\gamma\in G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$. We will denote the category of p-integral CLRs by $\mathbf{CLR}_K^p(G,X)$.

Lemma 4.6.9.

(1) *Every class in* $\mathbf{Cusp}_K(G,X)$ *has a representative in* $\mathbf{CLR}^p_K(G,X)$.
(2) *Fix* $\Phi \in \mathbf{CLR}^p_K(G,X)$*; then the Shimura datum* $\big(G_{\Phi,h}, F_\Phi\big)$ *is also unramified at* p.
(3) *If* (G,X) *is of Hodge type, then so is* $\big(G_{\Phi,h}, F_\Phi\big)$. *Moreover, for any prime* $v|p$ *of* $E = E(G,X)$*, the Shimura variety* Sh_{K_Φ} *over* $E(G,X)$ *has an integral canonical model* $\mathscr{S}_{K_\Phi}$ *over* $\mathscr{O}_{E,(v)}$.

Proof. The main point is that every parabolic sub-group of G extends uniquely to a parabolic sub-group of $G_{\mathbb{Z}_{(p)}}$. So every admissible parabolic sub-group $P \subset G$ extends to a parabolic sub-group $P_{\mathbb{Z}_{(p)}} \subset G_{\mathbb{Z}_{(p)}}$ with Levi quotient $L_{\mathbb{Z}_{(p)}}$. Furthermore, it follows from [Hör10, 1.6.9] that the normal sub-group $Q_P \subset P$ extends to a normal sub-group of $P_{\mathbb{Z}_{(p)}}$ with reductive image in $L_{\mathbb{Z}_{(p)}}$. From this, both (1) and (2) are clear.

When (G,X) is a Siegel Shimura datum, then so is $\big(G_{\Phi,h}, F_\Phi\big)$ (cf. 4.3.1). The first assertion of (3) is immediate from this. The second now follows from (4.6.4). □

Assumption 4.6.10. *From now on, all CLRs will be assumed to be* p*-integral, with respect to a* p*-integral structure on* (G,X,K) *that will be clear from context. All maps considered between such CLRs will also be in the category of* p*-integral CLRs.*

Definition 4.6.11. Let (G,X) be a Shimura datum, and let $K \subset G(\mathbb{A}_f)$ be a neat compact open sub-group. A p**-integral Hodge embedding** $\iota : (G,X,K) \hookrightarrow (\mathrm{GSp}(V), \mathrm{S}^\pm, K(n))$ into a Siegel Shimura datum consists of a p-integral embedding $\iota : (G,X) \hookrightarrow (\mathrm{GSp}(V), \mathrm{S}^\pm)$, a polarized lattice $V_{\mathbb{Z}} \subset V$ of discriminant d, and an $n \in \mathbb{Z}_{\geq 3}$ prime to pd such that:

- $K = K(n) \cap G(\mathbb{A}_f)$, where $K(n)$ is the level n sub-group of $\mathrm{GSp}\big(V_{\mathbb{A}_f}\big)$ associated with $V_{\mathbb{Z}}$.
- The map $\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K(n)}(\mathrm{GSp}, \mathrm{S}^\pm)$ is an embedding.

We will say that an admissible rppcd Σ for (G,X,K) is **associated** with the p-integral Hodge embedding $\iota : (G,X,K) \hookrightarrow (\mathrm{GSp}(V), \mathrm{S}^\pm, K(n))$ if there exists a smooth rppcd Σ' for $(\mathrm{GSp}, \mathrm{S}^\pm, K(n))$ such that Σ is induced from Σ'.

Assumption 4.6.12. *We will always assume from now on that* $K \subset G(\mathbb{A}_f)$ *is chosen so that there exists a* p*-integral embedding* $\iota : (G,X,K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^\pm, K(n))$. *Furthermore, any admissible rppcd* Σ *for* (G,X,K) *will be assumed to be associated with a* p*-integral Hodge embedding* $(G,X,K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^\pm, K(n))$. *This is not a very strong condition. By (4.1.5) and (4.2.20), we can always arrange this by shrinking* K *a little away from* p *and then by refining* Σ.

Theorem 4.6.13. *Assume that* $p > 2$. *Let* (G,X) *be a Shimura variety of Hodge type with reflex field* E, $K \subset G(\mathbb{A}_f)$ *a neat compact open, and* Σ *an admissible rppcd for* (G,X,K) *(recall our assumptions from (4.6.12). Fix* $v|p$*, a prime of* E *over* p. *Then there exists a flat integral model* $\mathscr{S}^\Sigma_K$ *of* $\mathrm{Sh}^\Sigma_K(G,X)$ *over* $\mathscr{O}_{E,(v)}$ *of the toroidal compactification* $\mathrm{Sh}^\Sigma_K(G,X)$ *from (4.2.19), containing the integral canonical model* $\mathscr{S}_K$ *of* $\mathrm{Sh}_K(G,X)$ *as an open dense sub-scheme, and satisfying the following properties:*

(1) *The boundary* $\mathbf{D}^\Sigma_K = \mathscr{S}^\Sigma_K \backslash \mathscr{S}_K$*, equipped with its reduced sub-scheme structure, is a relative effective Cartier divisor over* $\mathscr{O}_{E,(v)}$*, along which* $\mathscr{S}^\Sigma_K$ *has at worst toroidal singularities. In particular,* $\mathscr{S}^\Sigma_K$*, equipped with the log structure associated with* $\mathbf{D}^\Sigma_K$ *is log smooth over* $\mathscr{O}_{E,(v)}$.

(2) Σ *can be chosen such that* $\mathscr{S}^\Sigma_K$ *is smooth, projective, and such that* $\mathbf{D}^\Sigma_K$ *has normal crossings.*

(3) *There is a stratification by smooth $\mathscr{O}_{E,(v)}$-schemes*

$$\mathscr{S}_K^{\Sigma} = \bigsqcup_{[(\Phi,\sigma)]} \mathbf{Z}_{[(\Phi,\sigma)]},$$

where $[(\Phi,\sigma)]$ ranges over $\mathbf{Cusp}_K^{\Sigma}(G,X)$. This stratification is compatible with the stratification of $\mathrm{Sh}_K^{\Sigma}(G,X)$ described in (4.2.19)(3); in particular, $\mathbf{Z}_{[(\Phi,\sigma)]}$ lies in the closure of $\mathbf{Z}_{[(\Phi',\sigma')]}$ if and only if $[(\Phi',\sigma')]$ is a face of $[(\Phi,\sigma)]$.

(4) *For each $\Phi \in \mathbf{CLR}_K^p(G,X)$, $\mathscr{S}_{K_\Phi}$ is the integral canonical model of Sh_{K_Φ}, and $\mathbf{C}_\Phi$ is an abelian scheme over $\mathscr{S}_{K_\Phi}$.*

(5) *For each $[(\Phi,\sigma)]$ with representative (Φ,σ), $\mathbf{Z}_{[(\Phi,\sigma)]}$ is canonically isomorphic to the closed stratum in the twisted torus embedding $\mathbf{\Xi}_\Phi(\sigma)$ over the $\mathscr{S}_{K_\Phi}$-abelian scheme $\mathbf{C}_\Phi$. In fact, the completion $\mathfrak{X}_{[(\Phi,\sigma)]}$ of $\mathscr{S}_K^{\Sigma}$ along $\mathbf{Z}_{[(\Phi,\sigma)]}$ is canonically isomorphic to the completion of $\mathbf{\Xi}_\Phi(\sigma)$ along its closed stratum. This description is compatible with the analogous description of the strata found in (4.2.19)(4).*

Proof. Choose a p-integral Hodge embedding $(G,X,K) \hookrightarrow (\mathrm{GSp},\mathrm{S}^{\pm},K(n))$ into a Siegel Shimura datum with which Σ is associated. Take $\mathscr{S}_K^{\Sigma}$ to be the normalization of the Zariski closure of $\mathrm{Sh}_K^{\Sigma}(G,X)$ in $\mathscr{S}_{K(n)}^{\Sigma'}$.[4]

A good part of the theorem now follows from (4.4.7) and (4.4.13).

(3) was already shown in (4.4.13)(1). Consider (4): that $\mathscr{S}_{K_\Phi}$ is the integral canonical model for Sh_{K_Φ} is a consequence of (4.6.4) and (4.6.9). That $\mathbf{C}_\Phi$ is an abelian scheme over $\mathscr{S}_{K_\Phi}$ follows from (4.6.14) below. (5) and (1) are now immediate from (4.4.13)(3).

Finally, (2) follows from (4.3.7)(6) and (4.2.20). □

Lemma 4.6.14. *Let X be a healthy regular $\mathbb{Z}_{(p)}$-scheme (for example, X can be formally smooth over $\mathbb{Z}_{(p)}$), and let $A \to X$ be an abelian scheme. Suppose that $B_{\mathbb{Q}} \subset A_{\mathbb{Q}}$ is a finite map of abelian schemes over X. Let $B \to A$ be the normalization of the Zariski closure of the image of $B_{\mathbb{Q}}$ in $A_{\mathbb{Q}}$. Then B is an abelian scheme over X.*

Proof. From the usual Nerón-Ogg-Shafarevich good reduction criterion, we see that $B_{\mathbb{Q}}$ extends to an abelian scheme over all co-dimension 1 points of X. By healthy regularity, $B_{\mathbb{Q}}$ extends to an abelian scheme B' over X. By [FC90, I.2.7], the map $B_{\mathbb{Q}} \to A_{\mathbb{Q}}$ extends to a map $B' \to A$ over X. It is easy to check now that $B' = B$. □

Remark 4.6.15. As is clear from the proof, our methods will also work for $p = 2$ as soon as we know that the construction in (4.6.4) produces integral canonical (i.e. smooth) models.

Corollary 4.6.16. *Let (G,X,K), E and $v|p$ be as above. Then every geometric connected component of $\mathscr{S}_{K,k(v)}$ is the specialization of a unique geometric connected component of $\mathrm{Sh}_K(G,X)$.*

Proof. We now know that $\mathscr{S}_K$ admits a smooth compactification over $\mathscr{O}_{E,(v)}$, so the result follows from Zariski's connectedness theorem; cf. [DM69, 4.17]. □

Corollary 4.6.17. *Suppose that (G,X,K) is an unramified-at-p triple (cf. 4.6.7) of Hodge type, let $E = E(G,X)$ be its reflex field, and let $v|p$ be a prime of E. Then the integral canonical model $\mathscr{S}_K$ over $\mathscr{O}_{E,(v)}$ is proper if and only if $G/Z(G)$ is anisotropic.*

Proof. This is simply a special case of (4.4.7)(4). □

[4] We will see in (4.7.4) below that, when Σ is complete, $\mathscr{S}_K^{\Sigma}$ does not depend on the choice of p-integral embedding.

4.7. **Hecke action.** Here, we will show that the Hecke action on the tower of toroidal compactifications can be obtained entirely formally, once we are given integral canonical models and the Hecke action in characteristic 0.

4.7.1. Consider the following categories over $\mathscr{O}_{E,(v)}$: First, we take the category of pairs (T,D), where T is a proper normal flat $\mathscr{O}_{E,(v)}$-algebraic space, and $D \subset T$ is a sub-scheme that is a relative effective Cartier divisor over $\mathscr{O}_{E,(v)}$; the morphisms $(T,D) \to (T',D')$ are simply morphisms $T \to T'$ of $\mathscr{O}_{E,(v)}$-schemes that carry D into D'. Next, we consider triples (T°, T_E, D_E), where T° is a normal, flat $\mathscr{O}_{E,(v)}$-scheme, T_E is a proper normal algebraic space over E equipped with an open immersion $T^\circ \otimes E \hookrightarrow T_E$, and $D_E \subset E$ is an effective divisor whose complement is $T^\circ \otimes E$. A morphism $(T^\circ, T_E, D_E) \to (T^{',\circ}, T'_E, D'_E)$ is a pair of maps $T^\circ \to T^{',\circ}$ and $T_E \to T'_E$ of $\mathscr{O}_{E,(v)}$-schemes that agree on $T^\circ \otimes E$ and carry D_E into D'_E.

Lemma 4.7.2. *The natural functor $(T,D) \mapsto (T\backslash D, T \otimes E, D \otimes E)$ is fully faithful.*

Proof. Let k_v be the residue field of $\mathscr{O}_{E,(v)}$. Then, the lemma is a consequence of two facts:

(a) Given (T,D) as above, $D \otimes k_v$ has co-dimension 2 in T.
(b) Given a normal algebraic space T over $\mathscr{O}_{E,(v)}$, a closed sub-space $Z \hookrightarrow T$ of co-dimension at least 2, any map $T\backslash Z \to T'$ to a proper algebraic space T' over $\mathscr{O}_{E,(v)}$ extends uniquely to a map $T \to T'$.

The corresponding assertions for schemes over $\mathscr{O}_{E,(v)}$ are well-known, from which we can easily deduce the above for algebraic spaces as well. ☐

4.7.3. Suppose that there is a p-integral embedding $(\iota, g) : (G,X,K) \hookrightarrow (G',X',K')$ of unramified-at-p triples, as in (4.6.7) above, and suppose that (G',X') (and hence (G,X)) is of Hodge type. Let $E = E(G,X)$ be the reflex field of (G,X), let $v|p$ be a place of E and let E be the completion of E along v. Let $\mathscr{S}_K = \mathscr{S}_K(G,X)$ and $\mathscr{S}_{K'} = \mathscr{S}_{K'}(G',X')$ be the integral canonical models over $\mathscr{O}_{E,(v)}$ of $\mathrm{Sh}_K(G,X)$ and $\mathrm{Sh}_{K'}(G',X')$, respectively.[5] By (4.2.11), with every $\Phi \in \mathbf{CLR}^p_K(G,X)$, we can associated $\Phi' = (\iota,g)_*\Phi \in \mathbf{CLR}^p_K(G,X)$ such that $K_\Phi \subset K'_{\Phi'}$. In particular, there is a natural finite map $\mathscr{S}_{K_\Phi} \to \mathscr{S}_{K'_{\Phi'}}$ extending the embedding $\mathrm{Sh}_{K_\Phi}\big(G_{\Phi,h}, F_\Phi\big) \hookrightarrow \mathrm{Sh}_{K'_{\Phi'}}\big(G'_{\Phi',h}, F_{\Phi'}\big)$.

For any *complete* admissible rppcd Σ' (resp. Σ) for (G',X',K') (resp. (G,X,K)), over $\mathscr{O}_{E,(v)}$, there exists the proper toroidal compactifications $\mathscr{S}^{\Sigma'}_{K'}$ of $\mathscr{S}_{K'}$ and $\mathscr{S}^{\Sigma}_{K}$ of $\mathscr{S}_K$ from (4.6.13)

Proposition 4.7.4 (Hecke action)**.** *Suppose that Σ is a refinement of the complete admissible rppcd for (G,X,K) induced from Σ' along the embedding (ι,g). Then there is a unique proper map $[g]^{\Sigma,\Sigma'}_{K,K'} : \mathscr{S}^{\Sigma}_K \to \mathscr{S}^{\Sigma'}_{K'}$ with the following properties:*

(1) Its restriction to $\mathscr{S}_K$ agrees with the natural Hecke map $[g]_{K,K'} : \mathscr{S}_K \to \mathscr{S}_{K'}$ restricting to the corresponding Hecke map over E.

(2) Over E, $[g]^{\Sigma,\Sigma'}_{K,K'}$ agrees with the corresponding map defined in (4.2.21)(1).

(3) For any $[(\Phi,\sigma)] \in \mathbf{Cusp}^{\Sigma}_K(G,X)$, the stratum $\mathbf{Z}_{[(\Phi,\sigma)]}$ maps into the stratum $\mathbf{Z}_{[(\Phi',\sigma')]}$, where $[(\Phi',\sigma')] \in \mathbf{Cusp}^{\Sigma'}_{K'}(G',X')$ is determined in the following way: $\Phi' = (\iota,g)_*\Phi$ and $\sigma' \in \Sigma'_{\Phi'}$ is the minimal cone that contains σ.

(4) For (Φ,σ) and (Φ',σ') as in (3), the restriction $[g]_{[(\Phi,\sigma)]} : \mathbf{Z}_{[(\Phi,\sigma)]} \to \mathbf{Z}_{[(\Phi',\sigma')]}$ of $[g]^{\Sigma,\Sigma'}_{K,K'}$ can be described as follows: There exists a canonical homomorphism $\mathbf{C}_\Phi \to \mathbf{C}_{\Phi'}$ lying over the canonical map of integral canonical models $\mathscr{S}_{K_\Phi} \to \mathscr{S}_{K'_{\Phi'}}$, and lifting to an $\mathbf{E}_\Phi$-equivariant map $\Xi_\Phi \to \Xi_{\Phi'}$. Now, $[g]_{[(\Phi,\sigma)]}$ is isomorphic to the canonical map

[5] We are abusing notation here a little, since $\mathscr{S}_{K'}$ has up to now denoted the integral canonical model of $\mathrm{Sh}_{K'}(G',X')$ over the ring of integers of a completion of $E(G',X')$ (as opposed to $E(G,X)$).

between the closed stratum of $\Xi_{\Phi}(\sigma)$ *into the closed stratum of* $\Xi_{\Phi'}(\sigma')$. *Similarly, the induced map* $\widehat{[g]}_{[(\Phi,\sigma)]} : \mathfrak{X}_{[(\Phi,\sigma)]} \to \mathfrak{X}_{[(\Phi',\sigma')]}$ *on the completions along the strata is isomorphic to the canonical map between the completions of* $\Xi_{\Phi}(\sigma)$ *and* $\Xi_{\Phi'}(\sigma')$ *along their closed strata.*

In particular, the toroidal compactification $\mathscr{S}_K^{\Sigma}$ *does not depend on the choice of* p*-integral embedding of* (G,X) *into a Siegel Shimura datum.*

Proof. We first note that the map $[g]_{K',K}$ mentioned in (1) is the extension of the corresponding map in characteristic 0 (cf. 4.2.21) obtained via the extension property of the integral canonical model $\mathscr{S}_{K'}$ (or, rather, of the pro-scheme $\mathscr{S}_{K'_p}$; cf. 4.6.4). The existence and uniqueness of the map $[g]^{\Sigma',\Sigma}_{K',K}$ with properties (1) and (2) now follows from (4.7.2). Assertions (3) and (4) are immediate from their characteristic 0 counterparts (4.2.21)(2) and (4.2.21)(3). □

Remark 4.7.5. The functorial properties of our compactifications show that they agree with the compactifications of Lan [Lan08] when (G,X) is of PEL type.

4.8. **The minimal compactification.** With the results of § 4.6, we can now easily construct the minimal or Baily-Borel-Satake compactification of the integral canonical model $\mathscr{S}_K$ of a Shimura variety $\mathrm{Sh}_K(G,X)$ of Hodge type at a prime p where (G,X,K) is as in (4.6.13). We will follow the strategy in [FC90, §V.2], which is extended to the PEL case in [Lan08, §7.2]. Since the method here is not very different from that used in *loc. cit.*, our treatment will be somewhat compressed.

Definition 4.8.1. Suppose that there is a p-integral Hodge embedding $\iota : (G,X,K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm}, K(n))$. The **Hodge bundle** $\omega(\iota)$ over $\mathscr{S}_K$ associated with ι is the top exterior power of $\boldsymbol{V}_{\mathrm{dR}}(\iota)/F^1\boldsymbol{V}_{\mathrm{dR}}(\iota)$.

Suppose Σ is an admissible rppcd for (G,X,K) associated with the embedding ι. Then the **Hodge bundle** $\omega(\iota)^{\Sigma}$ over $\mathscr{S}_K^{\Sigma}$ is the top exterior power of $\boldsymbol{V}^{\Sigma}_{\mathrm{dR}}(\iota)/F^1\boldsymbol{V}^{\Sigma}_{\mathrm{dR}}(\iota)$.

Definition 4.8.2. Let S be an algebraic space over $\mathscr{O}_{E,(v)}$. Define an equivalence relation on line bundles on S as follows: We say that $\mathcal{L}_1 \sim_{\mathrm{Proj}} \mathcal{L}_2$ for line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over S if there exist a map $f : S' \to S$ of $\mathscr{O}_{E,(v)}$-schemes and integers $n,m \in \mathbb{Z}_{>0}$ such that:

- $f_*\mathscr{O}_{S'} = \mathscr{O}_S$ and $R^if_*\mathscr{O}_{S'} = 0$, for all $i > 1$.
- There exists an isomorphism of line bundles $f^*\mathcal{L}_1^{\otimes n} \xrightarrow{\simeq} f^*\mathcal{L}_2^{\otimes m}$ over S'.

We will denote the set of equivalence classes of line bundles over S under $\sim_{\mathrm{Proj}}$ by $\mathrm{Pic}(S)_{\mathbb{Z}}$.

Lemma 4.8.3. *Suppose that* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are two line bundles over* S *with* $\mathcal{L}_1 \sim_{\mathrm{Proj}} \mathcal{L}_2$. *Then the* $\mathscr{O}_{E,(v)}$*-schemes* $\mathrm{Proj}\Bigl(\bigoplus_k H^0(S,\mathcal{L}_1^{\otimes k})\Bigr)$ *and* $\mathrm{Proj}\Bigl(\bigoplus_k H^0(S,\mathcal{L}_2^{\otimes k})\Bigr)$ *are isomorphic.*

Proof. Let $f : S' \to S$ and $n,m \in \mathbb{Z}_{>0}$ be as in the definition of $\mathcal{L}_1 \sim_{\mathbb{Z}} \mathcal{L}_2$ above. Then, for $r = 1,2$, the projection formula shows

$$R^if_*f^*\mathcal{L}_r^{\otimes k} = \begin{cases} \mathcal{L}_r^{\otimes k}, \text{ if } i = 0; \\ 0 \qquad , \text{ otherwise.} \end{cases}$$

In particular:

$$\mathrm{Proj}\Bigl(\bigoplus_k H^0(S,\mathcal{L}_r^{\otimes k})\Bigr) = \mathrm{Proj}\Bigl(\bigoplus_k H^0(S',f^*\mathcal{L}_r^{\otimes k})\Bigr).$$

So we can replace S by S' and assume that $\mathcal{L}_1^{\otimes n} \simeq \mathcal{L}_2^{\otimes m}$. Now the result follows from [EGAII, 2.4.7]. □

Definition 4.8.4. For any class $[\mathcal{L}] \in \operatorname{Pic}(S)_{\mathbb{Z}}$, set

$$\mathbb{P}(S, [\mathcal{L}]) = \operatorname{Proj}\Big(\bigoplus_k H^0(S, \mathcal{L}^{\otimes k})\Big).$$

By (4.8.3) above, this $\mathscr{O}_{E,(v)}$-scheme does not depend on the choice of representative $\mathcal{L}$.

Lemma 4.8.5. *Let Σ be a complete admissible rppcd for (G, X, K) associated with two different p-integral embeddings $\iota_k : (G, X, K) \hookrightarrow (\mathrm{GSp}(V_k), \mathrm{S}_k^{\pm}, K(n_k))$, for $k = 1, 2$.*

(1) There exist a refinement Σ' of Σ and integers $r_1, r_2 \in \mathbb{Z}_{>0}$ such that the pull-backs of the line bundles $(\omega(\iota_1)^{\Sigma})^{\otimes r_1}$ and $(\omega(\iota_2)^{\Sigma})^{\otimes r_2}$ to $\mathscr{S}_K^{\Sigma'}$ are isomorphic.[6]

(2) The class $[\omega^{\Sigma}] := [\omega(\iota)^{\Sigma}] \in \operatorname{Pic}(\mathscr{S}_K^{\Sigma})_{\mathbb{Z}}$ is independent of the choice of ι. Moreover, it satisfies $\big([1]_K^{\Sigma',\Sigma}\big)^*[\omega^{\Sigma}] = [\omega^{\Sigma'}]$, for any refinement Σ' of Σ.

Proof. Let $V_{k,\mathbb{Z}_{(p)}} \subset V_k$, for $k = 1, 2$, be symplectic $\mathbb{Z}_{(p)}$-lattices giving the rise to the p-integral structure on $\mathrm{GSp}(V_k)$. Take $r_1, r_2 \in \mathbb{Z}_{>0}$ such that $V_{1,\mathbb{Z}_{(p)}}^{\oplus r_1}$ and $V_{2,\mathbb{Z}_{(p)}}^{\oplus r_2}$ are isomorphic as symplectic $\mathbb{Z}_{(p)}$-modules. Let $\widetilde{V}_{\mathbb{Z}_{(p)}}$ be this common symplectic $\mathbb{Z}_{(p)}$-module with associated symplectic $\mathbb{Q}$-space $\widetilde{V}$. Then there are natural p-integral embeddings of $(\mathrm{GSp}(V_k), \mathrm{S}_k^{\pm})$ $(k = 1, 2)$ into the Siegel Shimura datum $(\mathrm{GSp}(\widetilde{V}), \widetilde{\mathrm{S}}^{\pm})$.

Choose a polarized lattice $\widetilde{V}_{\mathbb{Z}} \subset \widetilde{V}$ of discriminant $\widetilde{d}$. Choose $\widetilde{n} \in \mathbb{Z}_{>0}$ prime to $p\widetilde{d}$ such that such that both $K(n_1)$ and $K(n_2)$ map into $K(\widetilde{n}) \subset \mathrm{GSp}(\widetilde{V})(\mathbb{A}_f)$. Refining Σ, if necessary, we can assume that we have an admissible rppcd $\widetilde{\Sigma}$ for $(\mathrm{GSp}(\widetilde{V}), \widetilde{\mathrm{S}}^{\pm}, K(\widetilde{n}))$ inducing admissible rppcds Σ_k for $(\mathrm{GSp}(V_k), \mathrm{S}_k^{\pm}, K(n_k))$ and such that Σ refines $\iota_k^*\Sigma_k$, for $k = 1, 2$. By (4.7.4), we then get a natural diagram

$$\begin{tikzcd}
 & \mathscr{S}_{K(n_1)}^{\Sigma_1} \arrow[dr, "j_1"] & \\
\mathscr{S}_K^{\Sigma} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \mathscr{S}_{K(\widetilde{n})}^{\widetilde{\Sigma}} \\
 & \mathscr{S}_{K(n_2)}^{\Sigma_2} \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Let ω^{Σ_k} (resp. $\widetilde{\omega}^{\widetilde{\Sigma}}$) be the Hodge bundle on $\mathscr{S}_{K(n_k)}^{\Sigma_k}$ (resp. $\mathscr{S}_{K(\widetilde{n})}^{\widetilde{\Sigma}}$). Then by construction there exist, for $k = 1, 2$, natural isomorphisms $j_k^*\widetilde{\omega}^{\widetilde{\Sigma}} \xrightarrow{\simeq} \big(\omega^{\Sigma_k}\big)^{\otimes r_k}$. This immediately gives us (1).

Now (2) is immediate from the following

Claim 4.8.6. Write f for the map $[1]_K^{\Sigma',\Sigma} : \mathscr{S}_K^{\Sigma'} \to \mathscr{S}_K^{\Sigma}$; then $f_*\mathscr{O}_{\mathscr{S}_K^{\Sigma'}} = \mathscr{O}_{\mathscr{S}_K^{\Sigma}}$ and $R^i f_*\mathscr{O}_{\mathscr{S}_K^{\Sigma'}} = 0$, for $i > 0$.

We proceed as in [FC90, V.1.2(b)]: This is a statement that is local on $\mathscr{S}_K^{\Sigma}$. The description in (4.7.4) of the restriction of f to the completion of $\mathscr{S}_K^{\Sigma'}$ along its strata further reduces us to the situation where f is an equivariant blow-up of a torus embedding, where one can write down an explicit Čech complex to compute the cohomology. $\square$

4.8.7. Pick $\Phi \in \mathbf{CLR}_K^p(G, X)$, and set[7]

$$\Delta_{\Phi} = \frac{P_{\Phi}(\mathbb{Q})^+ \bigcap Q_{\Phi}(\mathbb{A}_f) g_{\Phi} K g_{\Phi}^{-1}}{Q_{\Phi}(\mathbb{Q})^+} = \frac{P_{\Phi}(\mathbb{Z}_{(p)})^+ \bigcap Q_{\Phi}(\mathbb{A}_f) g_{\Phi} K g_{\Phi}^{-1}}{Q_{\Phi}(\mathbb{Z}_{(p)})^+}.$$

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[6] It is implied *post facto* by (4.8.11)(3) below that we can take $\Sigma' = \Sigma$.

[7] The super-script '+' means that we are looking at the sub-group of elements fixing the connected component X_{Φ}^+.

This is a quotient of the group $\Gamma_\Phi \subset \mathrm{Aut}\big(H_\Phi\big)$ defined in (4.2.9) and is the group denoted Δ_1 in [Pin90, 6.3]. Moreover, we have a canonical map $\Delta_\Phi \to \mathrm{Aut}\big(\mathscr{S}_{K_\Phi}\big)$ given as follows: Each γ in $P_\Phi\big(\mathbb{Z}_{(p)}\big)^+ \bigcap Q_\Phi\big(\mathbb{A}_f\big) g_\Phi K g_\Phi^{-1}$ determines an automorphism $\Phi \xrightarrow{\gamma} \Phi$. Just as in (4.2.18), this gives us an automorphism $[\gamma]$ of Sh_{K_Φ}, which, as one checks easily, depends only on the image of γ in Δ_Φ. Since the action of Δ_Φ on Sh_{K_Φ} is defined via prime-to-p Hecke operators, it extends naturally to an action of Δ_Φ on the integral canonical model $\mathscr{S}_{K_\Phi}$. As shown in [Pin90, 6.3], this action factors through a finite quotient of Δ_Φ.

Lemma 4.8.8.

(1) The quotient map $\mathscr{S}_{K_\Phi} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ is a Galois cover. In particular, $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ is smooth over $\mathscr{O}_{E,(v)}$.

(2) For every admissible Σ for (G, X, K) and every $[(\Phi_1, \sigma_1)] \in \mathbf{Cusp}_K^\Sigma(G, X)$ with $[\Phi_1] = [\Phi]$, there is a canonical smooth surjective map $\mathbf{Z}_{[(\Phi_1,\sigma_1)]} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$.

Proof. Choose a p-integral embedding $\iota : (G, X, K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^\pm, K(n))$, and let $\Phi' = \iota_*\Phi$, so that we have a natural map $i_{\Phi,\Phi'} : \mathscr{S}_{K_\Phi} \to \mathscr{S}_{K(n)_{\Phi'}}$ of integral canonical models over $\mathscr{O}_{E,(v)}$. In turn, this gives us a map $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi} \to \Delta_{\Phi'} \backslash \mathscr{S}_{K(n)_{\Phi'}}$ of their quotients. It follows from [Mor08, p. 8] that $\Delta_{\Phi'}$ acts trivially on $\mathscr{S}_{K(n)_{\Phi'}}$. So we in fact have a factorization

$$i_{\Phi,\Phi'} : \mathscr{S}_{K_\Phi} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi} \to \mathscr{S}_{K(n)_{\Phi'}}.$$

But $i_{\Phi,\Phi'}$ is unramified; indeed, the construction of Kisin [Kis10] identifies the complete local rings of $\mathscr{S}_{K_\Phi}$ with quotients of complete local rings of $\mathscr{S}_{K(n)_{\Phi'}}$. So the map $\mathscr{S}_{K_\Phi} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ is also unramified. This gives us (1).

For (2), we note first that $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ is independent of the choice of representative Φ in $[\Phi] = [\Phi_1]$: Indeed, if Φ_2 is another representative, then we can choose any map $\Phi \xrightarrow{\eta} \Phi_2$, which will produce an isomorphism $[\eta] : \mathscr{S}_{K_\Phi} \xrightarrow{\simeq} \mathscr{S}_{K_{\Phi_2}}$. Two such isomorphisms will differ by an automorphism $[\gamma] \in \Delta_\Phi$, so $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ and $\Delta_{\Phi_2} \backslash \mathscr{S}_{K_{\Phi_2}}$ are canonically isomorphic.

If we fix a representative of the form (Φ, σ) for $[(\Phi_1, \sigma_1)]$, then we have a canonical isomorphism $\mathbf{Z}_{[(\Phi_1,\sigma_1)]} \xrightarrow{\simeq} \mathbf{Z}_\Phi(\sigma)$, giving us a map $i_{\Phi,\sigma} : \mathbf{Z}_{[(\Phi_1,\sigma_1)]} \to \mathscr{S}_{K_\Phi}$. If we have an automorphism $\Phi \xrightarrow{\simeq} \Phi$ and $\sigma' \subset \Sigma_\Phi$ with $\gamma^*\sigma' = \sigma$, then, according to (4.2.19)(5), $i_{\Phi,\sigma'} = [\gamma] \circ i_{\Phi,\sigma}$. This shows that we have a canonical map $\mathbf{Z}_{[(\Phi_1,\sigma_1)]} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$; it is clearly smooth and surjective (use (1)). ∎

Proposition 4.8.9. *Let $\iota : (G, X, K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^\pm, K(n))$ be a p-integral Hodge embedding, and suppose that Σ is an admissible rppcd for (G, X, K) associated with ι. Let $\omega(\iota)^\Sigma$ be the Hodge bundle over $\mathscr{S}_K^\Sigma$ induced via ι, and denote by $\mathscr{S}_K^{\min}$ the projective $\mathscr{O}_{E,(v)}$-scheme $\mathbb{P}\big(\mathscr{S}_K^\Sigma, [\omega^\Sigma]\big)$.*

(1) A suitable power of $\omega(\iota)^\Sigma$ is generated by global sections. The map $f : \mathscr{S}_K^\Sigma \to \mathbb{P}^{r_0}_{\mathscr{O}_{E,(v)}}$ into projective space corresponding to the linear system attached to such a power of $\omega(\iota)^\Sigma$ has a Stein factorization $\mathscr{S}_K^\Sigma \xrightarrow{\oint^\Sigma} \mathscr{S}_K^{\min} \hookrightarrow \mathbb{P}^{r_0}_{\mathscr{O}_{E,(v)}}$.

(2) For any $[(\Phi, \sigma)] \in \mathbf{Cusp}_K^\Sigma(G, X)$ the restriction of $\oint^\Sigma$ to $\mathbf{Z}_{[(\Phi,\sigma)]}$ factors through the canonical smooth surjective map $\mathbf{Z}_{[(\Phi,\sigma)]} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ (cf. 4.8.8). The induced map from $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ to $\mathscr{S}_K^{\min}$ depends only on the class $[\Phi] \in \mathbf{Cusp}_K(G, X)$.

(3) If $[\Phi] \neq [\Phi']$ in $\mathbf{Cusp}_K(G, X)$, then the images of $\mathbf{Z}_{[(\Phi,\sigma)]}$ and $\mathbf{Z}_{[(\Phi',\sigma')]}$ in $\mathscr{S}_K^{\min}$ are disjoint.

(4) The map $\mathbf{Z}_{[(\Phi,\sigma)]} \to \Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$ is an isomorphism if and only if the unipotent radical $U_\Phi \subset P_\Phi$ has dimension at most 1.

Proof. As in [FC90, V.2.1], we can use [MB85, IX.2.1] to deduce the first part of (1). The second part follows from the argument in [Lan08, 7.2.3].

(2) is shown exactly as for the PEL case; cf. [Lan08, 7.2.3, esp. the discussion after 7.2.3.5].

We now show (3): Let $\overline{x} \to \mathscr{S}_K^{\min}$ be a geometric point in the intersection of the images of $\mathbf{Z}_{[(\Phi,\sigma)]}$ and $\mathbf{Z}_{[(\Phi',\sigma')]}$, and let C be a proper smooth connected curve over $k(\overline{x})$ mapping into $\left(\oint^{\Sigma}\right)^{-1}(\overline{x})$, and whose image intersects both $\mathbf{Z}_{[(\Phi,\sigma)]}$ and $\mathbf{Z}_{[(\Phi',\sigma')]}$. Suppose that the generic point of C maps into the stratum $\mathbf{Z}_{[(\widetilde{\Phi},\widetilde{\sigma})]} \subset \mathscr{S}_K^{\Sigma}$. Then C maps into the closure of $\mathbf{Z}_{[(\widetilde{\Phi},\widetilde{\sigma})]}$. From the description of the closure of strata in (4.6.13)(3), it follows that we have maps $\Phi \xrightarrow{\gamma} \widetilde{\Phi}$ and $\Phi' \xrightarrow{\gamma'} \widetilde{\Phi}$ in $\mathbf{CLR}_K(G,X)$. From [FC90, V.2.2], we see that the abelian part of the semi-abelian scheme induced over C is iso-trivial. The argument in [Lan08, 7.2.3.6] now shows that $\iota_*[\Phi] = \iota_*[\Phi'] = \iota_*[\widetilde{\Phi}]$ in $\mathbf{Cusp}_{K(n)}(\mathrm{GSp},\mathrm{S}^{\pm})$. This implies that $[\Phi] = [\widetilde{\Phi}] = [\Phi']$; cf. (4.2.11)(3).

For the map in (4) to be an isomorphism, we would need $\mathbf{C}_\Phi$ to be the trivial abelian scheme over $\mathscr{S}_{K_\Phi}$, which means that $U_\Phi = U_\Phi^{-2}$, and $\mathbf{E}_\Phi$ would have to be a torus of rank at most 1, which means that U_Φ^{-2} has dimension at most 1. So we find that U_Φ must have dimension at most 1.

Conversely, if U_Φ has dimension at most 1, then there are two possibilites: Either U_Φ is trivial, in which case we clearly have $\mathbf{\Xi}_\Phi = \mathscr{S}_{K_\Phi}$. The other possibility is that $\dim U_\Phi = 1$, in which case $\sigma = \mathbf{H}_\Phi$, and $\mathbf{\Xi}_\Phi(\sigma) \to \mathscr{S}_{K_\Phi}$ is isomorphic to the affine line over $\mathscr{S}_{K_\Phi}$, so that $\mathbf{Z}_\Phi(\sigma) \to \mathscr{S}_{K_\Phi}$ is again an isomorphism.

We now claim that, in both these cases, the conjugation action of $P_\Phi(\mathbb{Q})$ on $Q_\Phi(\mathbb{Q})$ is via inner automorphisms. This is clear when $P_\Phi = G$. When dim $U_\Phi = 1$, it follows from the classification of simple algebraic groups, that G^{ad} has a factor isomorphic to $\mathrm{PGL}_{2,\mathbb{Q}}$, and that P_Φ is the pre-image in G of a Borel sub-group of this factor. From this description, the claim is easily checked. It now follows that in both cases Δ_Φ acts trivially on $\mathscr{S}_{K_\Phi}$ and (4) is proved. $\square$

4.8.10. Fix $\Phi \in \mathbf{CLR}_K^p(G,X)$, and consider the tower $\mathbf{\Xi}_\Phi \to \mathbf{C}_\Phi \xrightarrow{\pi_\Phi} \mathscr{S}_{K_\Phi}$. The action of Δ_Φ on $\mathscr{S}_{K_\Phi}$ extends naturally to an action on the whole tower. This can be seen from the point of view of integral canonical models of mixed Shimura varieties (cf. [Hör10]).

As a $\mathbf{C}_\Phi$-scheme, we can write

$$\mathbf{\Xi}_\Phi = \underline{\mathrm{Spec}} \bigoplus_{\ell \in \mathbf{S}_\Phi} \Psi_\Phi(\ell),$$

where, for each ℓ, $\Psi_\Phi(\ell)$ is a line bundle over $\mathbf{C}_\Phi$. Set, for any $\ell \in \mathbf{S}_\Phi$, $\mathrm{FJ}_\Phi^{(\ell)} = \pi_{\Phi,*}\Psi(\ell)$: this is a coherent sheaf over $\mathscr{S}_{K_\Phi}$. For $\ell,\ell' \in \mathbf{S}_\Phi$, we have a natural 'multiplication' map $\mathrm{FJ}_\Phi^{(\ell)} \otimes_{\mathscr{O}_{E,(v)}} \mathrm{FJ}_\Phi^{(\ell')} \to \mathrm{FJ}_\Phi^{(\ell+\ell')}$, and we also have an identification $\mathrm{FJ}_\Phi^{(0)} = \mathscr{O}_{\mathscr{S}_{K_\Phi}}$. For any geometric point $\overline{x} \to \mathscr{S}_{K_\Phi}$, let $\widehat{\mathrm{FJ}}_{\Phi,\overline{x}}^{(\ell)}$ be the completion of the stalk of $\mathrm{FJ}_\Phi^{(\ell)}$ at $\overline{x}$. Also, let $\widehat{\mathfrak{m}}_{\overline{x}}$ be the maximal ideal of the completion of $\mathscr{S}_{K_\Phi}$ at $\overline{x}$. Let $\mathbf{H}_\Phi^\vee \subset \mathbf{S}_\Phi$ be the collection of elements that pair non-negatively with $\mathbf{H}_\Phi$. Consider $\prod_{\ell \in \mathbf{H}_\Phi^\vee} \widehat{\mathrm{FL}}_{\Phi,\overline{x}}^{(\ell)}$: this is an $\mathscr{O}_{E,(v)}$-algebra, and it is equipped with a maximal ideal $\widehat{\mathfrak{m}}_{\overline{x}} \times \prod_{0 \neq \ell \in \mathbf{H}_\Phi^\vee} \widehat{\mathrm{FL}}_{\Phi,\overline{x}}^{(\ell)}$, along which it is in fact complete. It is also naturally equipped with an action of the discrete automorphism group Δ_Φ.

Theorem 4.8.11. *Let (G,X,K) be as in (4.6.13). Then there exists a normal projective $\mathscr{O}_{E,(v)}$-scheme $\mathscr{S}_K^{\min}$ in which $\mathscr{S}_K$ embeds as a dense open sub-scheme, and which enjoys the following properties:*

(1) *For every complete admissible rppcd Σ for (G,X,K), we have an isomorphism of $\mathscr{O}_{E,(v)}$-schemes $\mathbb{P}\big(\mathscr{S}_K^{\Sigma},[\omega_K^{\Sigma}]\big) \xrightarrow{\simeq} \mathscr{S}_K^{\min}$.*

(2) *For any p-integral Hodge embedding $\iota : (G,X,K) \hookrightarrow (\mathrm{GSp},\mathrm{S}^{\pm},K(n))$, the Hodge bundle $\omega(\iota)$ extends to an ample line bundle $\omega(\iota)^{\min}$ over $\mathscr{S}_K^{\min}$. The class $[\omega^{\min}]$ of $\omega(\iota)^{\min}$ in $\mathrm{Pic}\big(\mathscr{S}_K^{\min}\big)_{\mathbb{Z}}$ is independent of the choice of ι. In fact, given a different p-integral*

embedding ι', there exist $r, s \in \mathbb{Z}_{>0}$ such that $(\omega(\iota)^{\min})^{\otimes r}$ and $(\omega(\iota')^{\min})^{\otimes s}$ are isomorphic.

(3) *For every Σ as in (1), there is a natural proper surjective map $\oint^{\Sigma} : \mathscr{S}_K^{\Sigma} \to \mathscr{S}_K^{\min}$ with geometrically connected fibers extending the identity on $\mathscr{S}_K$, such that $\left(\oint^{\Sigma}\right)^*[\omega^{\min}] = [\omega^{\Sigma}]$. In fact, if Σ is associated with a p-integral embedding ι, then $\left(\oint^{\Sigma}\right)^*\omega(\iota)^{\min} = \omega(\iota)^{\Sigma}$.*

(4) *$\oint^{\Sigma}$ is universal in the following sense: if $f : \mathscr{S}_K^{\Sigma} \to T$ is any other map to an $\mathscr{O}_{E,(v)}$-scheme T such that there is an ample line bundle $\mathcal{L}$ over T with $f^*\mathcal{L} \xrightarrow{\simeq} \left(\omega(\iota)^{\Sigma}\right)^{\otimes n}$, for some $n \in \mathbb{Z}_{>0}$, then f factors through $\oint^{\Sigma}$.*

(5) *There is a natural stratification*

$$\mathscr{S}_K^{\min} = \bigsqcup_{[\Phi] \in \mathbf{Cusp}_K(G,X)} \mathbf{Z}_{[\Phi]}$$

into locally closed smooth $\mathscr{O}_{E,(v)}$-sub-schemes. In this stratification, $\mathbf{Z}_{[\Phi']}$ is in the closure of $\mathbf{Z}_{[\Phi]}$ if and only if there is a map $\Phi' \xrightarrow{\gamma} \Phi$ in $\mathbf{CLR}_K^p(G,X)$.

(6) *Fix Φ in $\mathbf{CLR}_K^p(G,X)$. For every geometric point $\overline{x} \to \mathscr{S}_K^{\min}$ lying in the stratum $\mathbf{Z}_{[\Phi]}$, we have an isomorphism of complete local $\mathscr{O}_{E,(v)}$-algebras*

$$\widehat{\mathscr{O}}_{\mathscr{S}_K^{\min},\overline{x}} \xrightarrow{\simeq} \left(\prod_{\ell \in \mathbf{H}_\Phi^\vee} \widehat{\mathrm{FL}}^{(\ell)}_{\Phi,\overline{x}}\right)^{\Delta_\Phi}.$$

In particular, the natural map $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi} \to \mathbf{Z}_{[\Phi]}$ is an isomorphism.

(7) *The map $\oint^{\Sigma}$ is compatible with stratifications in the following sense: For any $[\Phi]$ in $\mathbf{Cusp}_K(G,X)$, the pre-image of $\mathbf{Z}_{[\Phi]}$ under $\oint^{\Sigma}$ is $\bigsqcup \mathbf{Z}_{[(\Phi,\sigma)]}$, where the disjoint union ranges over classes in $\mathbf{Cusp}_K^{\Sigma}(G,X)$ of the form $[(\Phi,\sigma)]$. Moreover, the induced map $\mathbf{Z}_{[(\Phi,\sigma)]} \to \mathbf{Z}_{[\Phi]}$ is, for any choice of representative (Φ,σ), isomorphic to the natural map from the closed stratum of $\mathbf{\Xi}_\Phi(\sigma)$ to $\Delta_\Phi \backslash \mathscr{S}_{K_\Phi}$: in particular, it is smooth and surjective.*

(8) *Let $\mathscr{S}_K^1 \subset \mathscr{S}_K^{\Sigma}$ be the pre-image of the complement in $\mathscr{S}_K^{\min}$ of the union of the strata of codimension at least 2. Then $\mathscr{S}_K^1$ maps isomorphically into $\mathscr{S}_K^{\min}$. Moreover, for any p-integral embedding $\iota : (G,X,K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm}, K(n))$ with which Σ is associated, and for every $k \in \mathbb{Z}_{>0}$, the natural map*

$$H^0\left(\mathscr{S}_K^{\Sigma}, \left(\omega(\iota)^{\Sigma}\right)^{\otimes k}\right) \to H^0\left(\mathscr{S}_K^{1}, \left(\omega(\iota)^{\Sigma}\right)^{\otimes k}\right)$$

is an isomorphism.

Proof. As mentioned earlier, everything here follows from arguments in [FC90, §V.2] and [Lan08, 7.2.3], so we allow ourselves to be somewhat terse.

As in (4.8.9), we can take $\mathscr{S}_K^{\min} = \mathbb{P}\left(\mathscr{S}_K^{\Sigma}, [\omega^{\Sigma}]\right)$, for any admissible rppcd Σ for (G,X,K). Then (4.8.5)(2), along with (4.8.6) and (4.8.3), shows that $\mathscr{S}_K^{\min}$ does not depend on the choice of Σ. We see from (4.8.9) that $\mathscr{S}_K^{\min}$ is projective. That it is in fact normal now follows from [Lan08, 7.2.3.1].

Denote by $\mathbf{Z}_{[\Phi]}$ the common image in $\mathscr{S}_K^{\min}$ under $\oint^{\Sigma}$ of all the strata of the form $\mathbf{Z}_{[(\Phi',\sigma')]}$, with $[\Phi'] = [\Phi]$. Then (4.8.9)(3) shows that the pre-image of $\mathbf{Z}_{[\Phi]}$ is $\bigsqcup_{[\Phi']=[\Phi]} \mathbf{Z}_{[(\Phi',\sigma')]}$. Assertions (5), (6) and (7) now easily follow from the arguments in [Lan08, 7.2.3.5,7,13]. In

particular, we find that $\mathscr{S}_K$ (the union of the strata corresponding to the improper cusp labels) maps isomorphically onto an open dense sub-scheme of $\mathscr{S}_K^{\min}$.

We still have to show assertions (2), (3) and (4). The first part of (3) is clear: there is at most one map $\oint^{\Sigma} : \mathscr{S}_K^{\Sigma} \to \mathscr{S}_K^{\min}$ extending the identity on $\mathscr{S}_K$, and we have seen in (4.8.9) that there is at least one such map. As for (2), the arguments of [Lan08, 7.2.4.1] show that we can take $\omega^{\min}(\iota)$ to be the class of $\left(\oint^{\Sigma}\right)_* \omega(\iota)^{\Sigma}$ (the main point is to show that this latter sheaf is a line bundle), for any Σ associated with ι. That $\left(\oint^{\Sigma}\right)^*[\omega^{\min}] = [\omega^{\Sigma}]$ is now clear from (4.8.5).

Assertion (4) is clear from the description of $\oint^{\Sigma}$ as a Stein factorization above.

As for (8), the first part follows from (4.8.9)(4), which shows that $\mathscr{S}_K^1$ maps bijectively onto its image. It maps isomorphically because the map $\oint^{\Sigma}$ is proper and because $\mathscr{S}_K^{\min}$ is normal. The second part about the extension of sections of the Hodge bundle is now deduced in the standard way; cf. [Lan08, 7.2.4.8]. □

Remark 4.8.12. When G^{ad} does not admit $\mathrm{PGL}_{2,\mathbb{Q}}$ as a simple factor, $\mathscr{S}_K^1 = \mathscr{S}_K$, and (4.8.11)(8) gives us **Koecher's principle** for sections of powers of the Hodge bundle $\omega(\iota)$.

Remark 4.8.13. It is possible now to repeat the arguments in [Lan08, §7.3] to show that, whenever the rppcd Σ admits a polarization function (cf. [Lan08, 7.3.1.1]), $\mathscr{S}_K^{\Sigma}$ will be the normalization of the blow-up of $\mathscr{S}_K^{\min}$ along a very explicit sheaf of ideals: in particular, it will be a projective scheme. We do not go into the details here, since our construction already gives projective toroidal compactifications of $\mathscr{S}_K^{\Sigma}$ for free: We only have to choose a p-integral embedding $(G, X, K) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm}, K(n))$, and then we are free to choose any projective, smooth admissible rppcd Σ' for the Siegel Shimura datum. The induced rppcd Σ for (G, X, K) will be automatically projective, and the compactification $\mathscr{S}_K^{\Sigma}$ will also be projective, since it is finite over the projective compactification $\mathscr{S}_{K(n)}^{\Sigma'}$.

Proposition 4.8.14 (Hecke action)**.** *Suppose that we have a p-integral embedding $(\iota, g) : (G, X, K) \hookrightarrow (G', X', K')$ of unramified-at-p triples of Hodge type, as in (4.7.4). Let $E = E(G, X)$, and let $v|p$ be a finite prime of E. Then there is a unique map $[g]_{K,K'}^{\min} : \mathscr{S}_K^{\min} \to \mathscr{S}_{K'}^{\min}$ of $\mathscr{O}_{E,(v)}$-schemes extending the Hecke map $[g]_{K',K} : \mathscr{S}_K \to \mathscr{S}_{K'}$ and enjoying the following properties:*

(1) For every $[\Phi] \in \mathbf{Cusp}_K(G, X)$, $[g]_{K,K'}^{\min}$ maps the stratum $\mathbf{Z}_{[\Phi]}$ into the stratum $\mathbf{Z}_{[\Phi']}$, where $[\Phi'] = (\iota, g)_*[\Phi]$.

(2) For every $[\Phi] \in \mathbf{Cusp}_K(G, X)$, the restriction of $[g]_{K,K'}^{\min}$ to $\mathbf{Z}_{[\Phi]}$ is isomorphic to the natural map $\Delta_{\Phi} \backslash \mathscr{S}_{K_{\Phi}} \to \Delta_{\Phi'} \backslash \mathscr{S}_{K'_{\Phi'}}$ of quotients of integral canonical models, where Φ is a representative of $[\Phi]$ and $\Phi' = (\iota, g)_* \Phi$.

(3) Given a p-integral embedding $\iota_1' : (G', X', K') \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm}, K(n))$, we have

$$\left([g]_{K,K'}^{\min}\right)^* \omega(\iota_1')^{\min} \xrightarrow{\simeq} \omega(\iota_1)^{\min}.$$

Proof. The uniqueness of such an extension is clear. We only have to show its existence. Choose any admissible rppcd Σ' for (G', X', K'), and let Σ be the induced admissible rppcd for (G, X, K). We claim that we have a commuting diagram

$$\begin{tikzcd} \mathscr{S}_K^{\Sigma} \arrow[r, "{[g]_{K,K'}^{\Sigma,\Sigma'}}"] \arrow[d, "\oint_K^{\Sigma}"'] & \mathscr{S}_{K'}^{\Sigma'} \arrow[d, "\oint_{K'}^{\Sigma'}"] \\ \mathscr{S}_K^{\min} \arrow[r, "{[g]_{K,K'}^{\min}}"'] & \mathscr{S}_{K'}^{\min}. \end{tikzcd}$$

Here, $[g]^{\Sigma,\Sigma'}_{K,K'}$ is the Hecke map from (4.7.4). This claim is enough, since all the claimed properties of $[g]^{\min}_{K,K'}$ will now follow easily from the corresponding properties of $[g]^{\Sigma,\Sigma'}_{K,K'}$.

To prove the claim, we have to show that the composition $\oint^{\Sigma'}_{K'} \circ [g]^{\Sigma,\Sigma'}_{K,K'}$ factors through $\oint^{\Sigma}_{K}$, but this follows from (4.8.11)(4). □

References

[Ale02] Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708, DOI 10.2307/3062130. MR1923963 (2003g:14059)

[AN99] Valery Alexeev and Iku Nakamura, *On Mumford's construction of degenerating abelian varieties*, Tohoku Math. J. (2) **51** (1999), no. 3, 399–420, DOI 10.2748/tmj/1178224770. MR1707764 (2001g:14013)

[And90] Yves André, *p-adic Betti lattices*, p-adic analysis (Trento, 1989), Lecture Notes in Math., vol. 1454, Springer, Berlin, 1990, pp. 23–63, DOI 10.1007/BFb0091133, (to appear in print). MR1094846 (92c:14015)

[ABV05] Fabrizio Andreatta and Luca Barbieri-Viale, *Crystalline realizations of 1-motives*, Math. Ann. **331** (2005), no. 1, 111–172, DOI 10.1007/s00208-004-0576-4. MR2107442 (2005h:14047)

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. With the collaboration of Peter Scholze. MR2590897

[BB66] Walter L. Baily Jr. and Armand Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528. MR0216035 (35 #6870)

[Ber04] Laurent Berger, *An introduction to the theory of p-adic representations*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 255–292. MR2023292 (2005h:11265)

[BBM82] Pierre Berthelot, Lawrence Breen, and William Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982 (French). MR667344 (85k:14023)

[BO78] Pierre Berthelot and Arthur Ogus, *Notes on crystalline cohomology*, Princeton University Press, Princeton, N.J., 1978. MR0491705 (58 #10908)

[BO83] P. Berthelot and A. Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), no. 2, 159–199, DOI 10.1007/BF01389319. MR700767 (85e:14025)

[Bla94] Don Blasius, *A p-adic property of Hodge classes on abelian varieties*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 293–308. MR1265557 (95j:14022)

[BHC62] Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535. MR0147566 (26 #5081)

[Bre97] Christophe Breuil, *Représentations p-adiques semi-stables et transversalité de Griffiths*, Math. Ann. **307** (1997), no. 2, 191–224, DOI 10.1007/s002080050031 (French). MR1428871 (98b:14016)

[Bre99] Christophe Breuil, *Représentations semi-stables et modules fortement divisibles*, Invent. Math. **136** (1999), no. 1, 89–122, DOI 10.1007/s002220050305 (French). MR1681105 (2000c:14024)

[Bre00] Christophe Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549. MR1804530 (2001k:14087)

[Bry83] Jean-Luc Brylinski, "1*-motifs" et formes automorphes (théorie arithmétique des domaines de Siegel)*, Conference on automorphic theory (Dijon, 1981), Publ. Math. Univ. Paris VII, vol. 15, Univ. Paris VII, Paris, 1983, pp. 43–106 (French). MR723182 (85g:11047)

[Cha90] C.-L. Chai, *Arithmetic minimal compactification of the Hilbert-Blumenthal moduli spaces*, Ann. of Math. (2) **131** (1990), no. 3, 541–554, DOI 10.2307/1971469. MR1053489 (91i:11063)

[CI99] Robert Coleman and Adrian Iovita, *The Frobenius and monodromy operators for curves and abelian varieties*, Duke Math. J. **97** (1999), no. 1, 171–215, DOI 10.1215/S0012-7094-99-09708-9. MR1682268 (2000e:14023)

[DOR10] Jean-François Dat, Sascha Orlik, and Michael Rapoport, *Period domains over finite and p-adic fields*, Cambridge Tracts in Mathematics, vol. 183, Cambridge University Press, Cambridge, 2010. MR2676072

[Del71] Pierre Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Springer, Berlin, 1971, pp. 123–165. Lecture Notes in Math., Vol. 244 (French). MR0498581 (58 #16675)

[Del72] Pierre Deligne, *La conjecture de Weil pour les surfaces $K3$*, Invent. Math. **15** (1972), 206–226 (French). MR0296076 (45 #5137)

[Del74] Pierre Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77 (French). MR0498552 (58 #16653b)

[Del79] Pierre Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289 (French). MR546620 (81i:10032)

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR654325 (84m:14046)

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109. MR0262240 (41 #6850)

[DP94] Pierre Deligne and Georgios Pappas, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*, Compositio Math. **90** (1994), no. 1, 59–79 (French). MR1266495 (95a:11041)

[Fal89] Gerd Faltings, *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80. MR1463696 (98k:14025)

[Fal99] Gerd Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. **12** (1999), no. 1, 117–144, DOI 10.1090/S0894-0347-99-00273-8. MR1618483 (99e:14022)

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. MR1083353 (92d:14036)

[Fon94a] Jean-Marc Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111 (French). With an appendix by Pierre Colmez; Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293971 (95k:11086)

[Fon94b] Jean-Marc Fontaine, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113–184 (French). With an appendix by Pierre Colmez; Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293972 (95g:14024)

[Gör01] Ulrich Görtz, *On the flatness of models of certain Shimura varieties of PEL-type*, Math. Ann. **321** (2001), no. 3, 689–727, DOI 10.1007/s002080100250. MR1871975 (2002k:14034)

[Gör03] Ulrich Görtz, *On the flatness of local models for the symplectic group*, Adv. Math. **176** (2003), no. 1, 89–115, DOI 10.1016/S0001-8708(02)00062-2. MR1978342 (2004d:14023)

[EGAII] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. **8** (1961), 222. MR0217084 (36 #177b)

[EGAIV2] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231 (French). MR0199181 (33 #7330)

[EGAIV3] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966), 255. MR0217086 (36 #178)

[SGA7I] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, Berlin, 1972 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I); Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. MR0354656 (50 #7134)

[Har89] Michael Harris, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3) **59** (1989), no. 1, 1–22, DOI 10.1112/plms/s3-59.1.1. MR997249 (90h:11048)

[Hör10] Fritz Hörmann, *The arithmetic volume of Shimura varieties of orthogonal type*, 2010. Thesis (Ph.D.)–Humboldt-University of Berlin.

[HK94] Osamu Hyodo and Kazuya Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, Astérisque **223** (1994), 221–268. Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293974 (95k:14034)

[KKN08a] Takeshi Kajiwara, Kazuya Kato, and Chikara Nakayama, *Logarithmic abelian varieties. I. Complex analytic theory*, J. Math. Sci. Univ. Tokyo **15** (2008), no. 1, 69–193. MR2422590 (2009f:14086)

[KKN08b] Takeshi Kajiwara, Kazuya Kato, and Chikara Nakayama, *Logarithmic abelian varieties*, Nagoya Math. J. **189** (2008), 63–138. MR2396584 (2009d:14061)

[Kat89] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224. MR1463703 (99b:14020)

[Kata] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie II*, Unpublished.

[Katb] Kazuya Kato, *Logarithmic Dieudonné theory*, Unpublished.

[KT03] Kazuya Kato and Fabien Trihan, *On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$*, Invent. Math. **153** (2003), no. 3, 537–592, DOI 10.1007/s00222-003-0299-2. MR2000469 (2004h:11058)

[KU09] Kazuya Kato and Sampei Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, vol. 169, Princeton University Press, Princeton, NJ, 2009. MR2465224 (2009m:14012)

[Kat81] N. Katz, *Serre-Tate local moduli*, Algebraic surfaces (Orsay, 1976), Lecture Notes in Math., vol. 868, Springer, Berlin, 1981, pp. 138–202. MR638600 (83k:14039b)

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973. MR0335518 (49 #299)

[Kim11] Wansu Kim, *Classification of p-divisible groups over 2-adic discrete valuation rings* (2011), available at `https://dl.dropbox.com/u/4296532/2adicKisin.pdf`. Preprint.

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496. MR2263197 (2007j:11163)

[Kis09a] Mark Kisin, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180, DOI 10.4007/annals.2009.170.1085. MR2600871

[Kis09b] Mark Kisin, *Integral canonical models of Shimura varieties*, J. Théor. Nombres Bordeaux **21** (2009), no. 2, 301–312 (English, with English and French summaries). MR2541427 (2010i:11087)

[Kis10] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012, DOI 10.1090/S0894-0347-10-00667-3. MR2669706 (2011j:11109)

[Kot90] Robert E. Kottwitz, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209. MR1044820 (92b:11038)

[Kot92] Robert E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444, DOI 10.2307/2152772. MR1124982 (93a:11053)

[KS67] Michio Kuga and Ichirô Satake, *Abelian varieties attached to polarized K_3-surfaces*, Math. Ann. **169** (1967), 239–242. MR0210717 (35 #1603)

[Kün98] Klaus Künnemann, *Projective regular models for abelian varieties, semistable reduction, and the height pairing*, Duke Math. J. **95** (1998), no. 1, 161–212, DOI 10.1215/S0012-7094-98-09505-9. MR1646554 (99m:14043)

[Kud04] Stephen S. Kudla, *Special cycles and derivatives of Eisenstein series*, Heegner points and Rankin L-series, Mat. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 243–270.

[Lan08] Kai-Wen Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Harvard University. MR2711676

[Lan10a] Kai-Wen Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, J. Reine Angew. Math., to appear (2010).

[Lan10b] Kai-Wen Lan, *Toroidal compactifications of PEL-type Kuga families* (2010). preprint.

[Lan10c] Kai-Wen Lan, *Elevators for degenerations of PEL structures* (2010). preprint.

[Lan76] Robert P. Langlands, *Some contemporary problems with origins in the Jugendtraum*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), Amer. Math. Soc., Providence, R. I., 1976, pp. 401–418. MR0437500 (55 #10426)

[Lar92] Michael J. Larsen, *Arithmetic compactification of some Shimura surfaces*, The zeta functions of Picard modular surfaces, Univ. Montréal, Montreal, QC, 1992, pp. 31–45. MR1155225 (93d:14037)

[Lau12] Eike Lau, *A relation between Dieudonne displays and crystalline Dieudonne theory* (2012), available at `http://arxiv.org/abs/1006.2720`. Preprint.

[Lee12] Dong Uk Lee, *On a conjecture of Y. Morita*, Bull. Lond. Math. Soc. **44** (2012), no. 5, 861–870, DOI 10.1112/blms/bdr104.

[Liu11] Tong Liu, *The correspondence between Barsotti-Tate groups and Kisin modules when p=2* (2011), available at `http://www.math.purdue.edu/~tongliu/pub/2BT.pdf`. Preprint.

[Lod07] Rémi S. Lodh, *Galois cohomology of Fontaine rings*, 2007. Dissertation (Ph.D.)–Bonn University.

[MS11] Keerthi Shyam Madapusi Sampath, *Toroidal compactifications of integral canonical models of Shimura varieties of Hodge type*, Ph. D. Thesis, University of Chicago, 2011.

[MP12a] Keerthi Madapusi Pera, *Regular integral models for Spin Shimura varieties*, In preparation (2012).

[MP12b] Keerthi Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, In preparation (2012).

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461 (90i:13001)

[Mes72] William Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, Berlin, 1972. MR0347836 (50 #337)

[Mil90] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 283–414. MR1044823 (91a:11027)

[Mil94] J. S. Milne, *Shimura varieties and motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 447–523. MR1265562 (95c:11076)

[Mil92] James S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces, Univ. Montréal, Montreal, QC, 1992, pp. 151–253. MR1155229 (94g:11041)

[Moo98] Ben Moonen, *Models of Shimura varieties in mixed characteristics*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 267–350. MR1696489 (2000e:11077)

[Mor08] Sophie Morel, *Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel*, J. Amer. Math. Soc. **21** (2008), no. 1, 23–61 (electronic), DOI 10.1090/S0894-0347-06-00538-8 (French). MR2350050 (2008i:11070)

[Mor10] Sophie Morel, *The intersection complex as a weight truncation and an application to Shimura varieties*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 312–334. MR2827798

[MB85] Laurent Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque **129** (1985), 266 (French, with English summary). MR797982 (87j:14069)

[Mor75] Yasuo Morita, *On potential good reduction of abelian varieties*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **22** (1975), no. 3, 437–447. MR0404269 (53 #8072)

[MT62] G. D. Mostow and T. Tamagawa, *On the compactness of arithmetically defined homogeneous spaces*, Ann. of Math. (2) **76** (1962), 446–463. MR0141672 (25 #5069)

[Mum69] David Mumford, *Bi-extensions of formal groups*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 307–322. MR0257089 (41 #1743)

[Nam80] Yukihiko Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Mathematics, vol. 812, Springer, Berlin, 1980. MR584625 (82a:32034)

[Niz08] Wiesława Nizioł, *K-theory of log-schemes. I*, Doc. Math. **13** (2008), 505–551. MR2452875

[Ols08] Martin C. Olsson, *Compactifying moduli spaces for abelian varieties*, Lecture Notes in Mathematics, vol. 1958, Springer-Verlag, Berlin, 2008. MR2446415 (2009h:14072)

[Pap00] Georgios Pappas, *On the arithmetic moduli schemes of PEL Shimura varieties*, J. Algebraic Geom. **9** (2000), no. 3, 577–605. MR1752014 (2001g:14042)

[PZ12] Georgios Pappas and Xinwen Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz* (2012), available at `http://arxiv.org/abs/1110.5588`. Preprint.

[Pau04] Frédéric Paugam, *Galois representations, Mumford-Tate groups and good reduction of abelian varieties*, Math. Ann. **329** (2004), no. 1, 119–160, DOI 10.1007/s00208-004-0514-5. MR2052871 (2005e:11072)

[Pin90] Richard Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Mathematische Schriften [Bonn Mathematical Publications], 209, Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989. MR1128753 (92h:11054)

[PY06] Gopal Prasad and Jiu-Kang Yu, *On quasi-reductive group schemes*, J. Algebraic Geom. **15** (2006), no. 3, 507–549, DOI 10.1090/S1056-3911-06-00422-X. With an appendix by Brian Conrad. MR2219847 (2007c:14047)

[Rap78] M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compositio Math. **36** (1978), no. 3, 255–335 (French). MR515050 (80j:14009)

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR1393439 (97f:14023)

[Ray71] Michel Raynaud, *Variétés abéliennes et géométrie rigide*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 473–477. MR0427326 (55 #360)

[SR72] Neantro Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag, Berlin, 1972 (French). MR0338002 (49 #2769)

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517. MR0236190 (38 #4488)

[Str10] Benoît Stroh, *Compactification de variétés de Siegel aux places de mauvaise réduction*, Bull. Soc. Math. France **138** (2010), no. 2, 259–315 (French, with English and French summaries). MR2679041

[Vas99] Adrian Vasiu, *Integral canonical models of Shimura varieties of preabelian type*, Asian J. Math. **3** (1999), no. 2, 401–518. MR1796512 (2002b:11087)

[Vas08] Adrian Vasiu, *Projective integral models of Shimura varieties of Hodge type with compact factors*, J. Reine Angew. Math. **618** (2008), 51–75, DOI 10.1515/CRELLE.2008.033. MR2404746 (2009c:11091)

[Vas12a] Adrian Vasiu, *Integral models in unramified mixed characteristic (0,2) of hermitian orthogonal Shimura varieties of PEL type, Part I* (2012), available at `http://arxiv.org/abs/math/0307205`. Preprint.

[Vas12b] Adrian Vasiu, *Integral models in unramified mixed characteristic (0,2) of hermitian orthogonal Shimura varieties of PEL type, Part II* (2012), available at `http://arxiv.org/abs/math/0606698`. Preprint.

[Vas12c] Adrian Vasiu, *Good Reductions of Shimura Varieties of Hodge Type in Arbitrary Unramified Mixed Characteristic, Part I* (2012), available at `http://arxiv.org/abs/0707.1668`. Preprint.

[Vas12d] Adrian Vasiu, *Good Reductions of Shimura Varieties of Hodge Type in Arbitrary Unramified Mixed Characteristic, Part II* (2012), available at `http://arxiv.org/abs/0712.1572`. Preprint.

[VZ10] Adrian Vasiu and Thomas Zink, *Purity results for p-divisible groups and abelian schemes over regular bases of mixed characteristic*, Doc. Math. **15** (2010), 571–599. MR2679067

[Vol03] Vadim Vologodsky, *Hodge structure on the fundamental group and its application to p-adic integration*, Mosc. Math. J. **3** (2003), no. 1, 205–247, 260 (English, with English and Russian summaries). MR1996809 (2004h:14019)

[Win94] Jean-Pierre Wintenberger, *Théorème de comparaison p-adique pour les schémas abéliens. I. Construction de l'accouplement de périodes*, Astérisque **223** (1994), 349–397 (French). Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293978 (96d:14019)

[Zar77] Ju. G. Zarhin, *Endomorphisms of abelian varieties and points of finite order in characteristic P*, Mat. Zametki **21** (1977), no. 6, 737–744 (Russian). MR0485893 (58 #5692)

[Zar85] Yu. G. Zarhin, *A finiteness theorem for unpolarized abelian varieties over number fields with prescribed places of bad reduction*, Invent. Math. **79** (1985), no. 2, 309–321, DOI 10.1007/BF01388976. MR778130 (86d:14041)

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